(Real time) Wilson loops and AdS/CFT

Ville Keränen
Nordita & University of Iceland

Introduction

- N=4 Super-Yang-Mills

$$S = -\frac{1}{g^2}\int d^4x \,\mathrm{Tr}\Big(\frac{1}{2}F_{\mu\nu}^2 + (D_\mu \Phi^I)^2 - \left[\Phi^I, \Phi^J\right]^2 + \mathrm{Fermions}\Big)$$

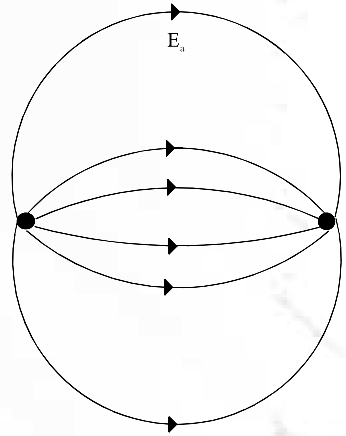
- Wilson loops provide good gauge invariant observables.
- They create flux tubes of gluon fields → Dual to open strings in AdS
- Motivations:

  - QCD flux tubes; N=4 SYM gives a calculable toy model for flux tube dynamics

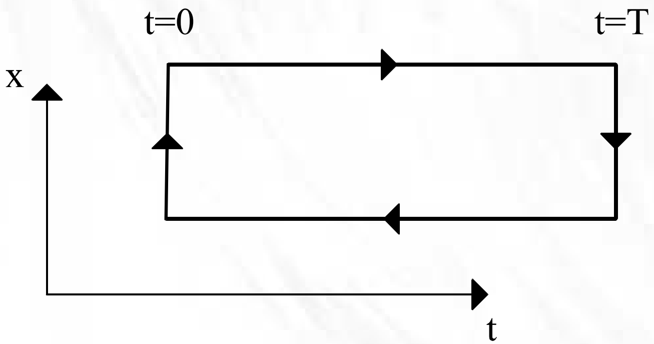
  - Study Wilson loops to obtain a better understanding of AdS/CFT (emergence of spacetime, black holes etc.)

Wilson loops in N=4 SYM

- Path integral in the presence of a loop of charge in the fundamental representation



$$"S \to S + \int d^4x A_\mu j^\mu" \qquad j^\mu = \frac{dx^\mu}{d\tau}\delta^3(x - x(\tau))$$



$$\langle W \rangle = \frac{1}{N}\int [dAd\Phi d\Psi] e^{iS} \mathrm{Tr} P e^{i \oint dx^\mu A_\mu}$$

- According to the Gauss' law the "quarks" are connected by a tube of gluon flux (like a dipole in electrodynamics)
- One can for example read off the energy eigenvalues of the flux tube with the lowest energy state being the quark anti-quark potential

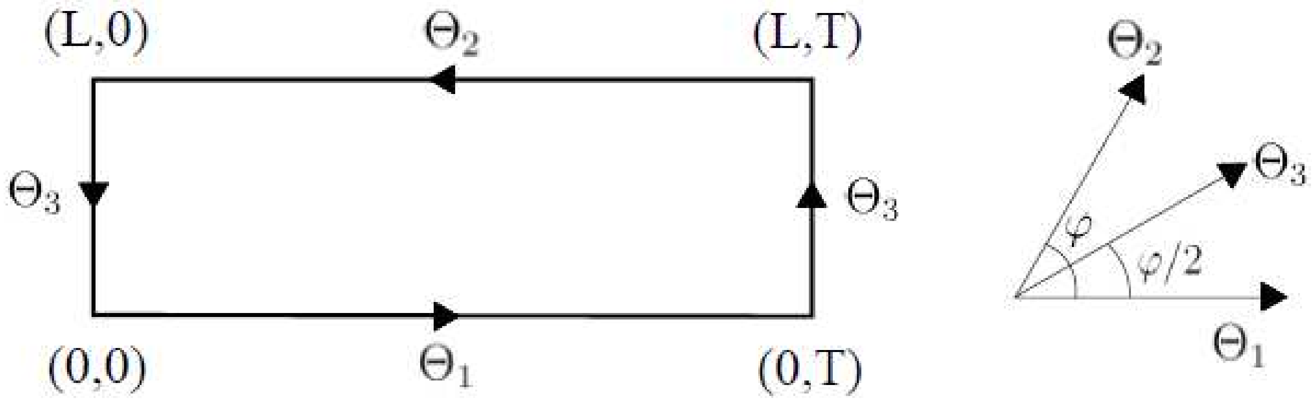
$$\langle W \rangle = \sum_n c_n e^{-iE_n T}$$

Wilson loops in N=4 SYM

- We consider the generalized Wilson loop [Maldacena]

$$\langle W(C)\rangle = \frac{1}{N}\langle \mathrm{Tr} P e^{i\int_C d\tau\,(A_\mu \dot{x}^\mu - \Phi_i \theta_i \sqrt{-\dot{x}^2})}\rangle$$

- The external "quarks" couple to the scalar fields and thus, have orientation in the SO(6)
- Θ Is a unit vector in S^5

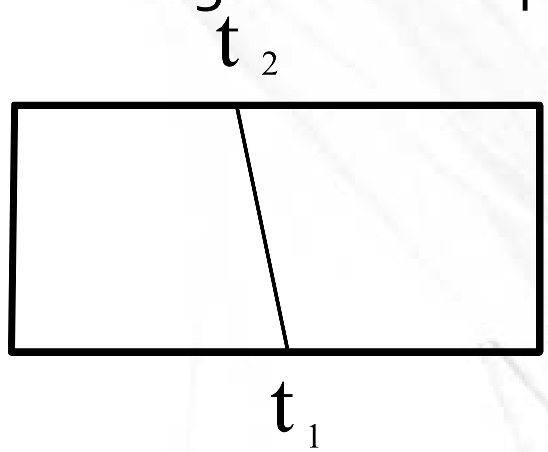


How to calculate them in PT

- Recipe: expand the exponential and compute the correlation functions order by order

$$\langle W \rangle \approx 1 - g^2 N \int_0^T d\tau_1 d\tau_2 \Big(\langle T A(\tau_1, 0) A(\tau_2, L) \rangle + \Theta_1 \cdot \Theta_2 \langle T \Phi(\tau_1, 0) \Phi(\tau_2, L) \rangle \Big) + \ldots$$

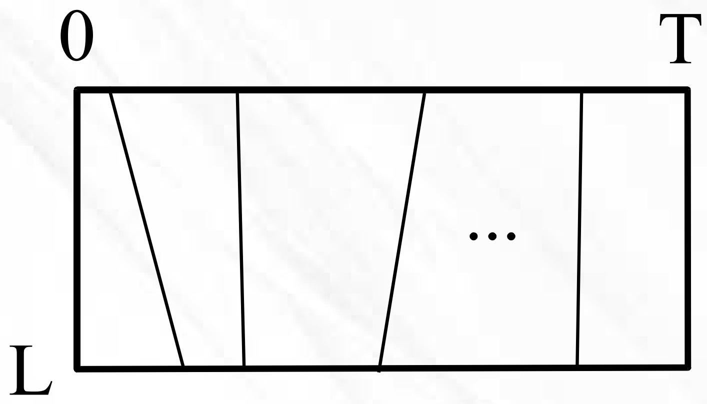
- Wilson loop obtained by summing all Feynman diagrams with external legs ending on the loop
- Gluon exchanges come in powers of $\lambda = g^2 N$
- Scalar exhanges come in powers of $\hat{\lambda} = \Theta_1 \cdot \Theta_2 \lambda$



A scaling limit

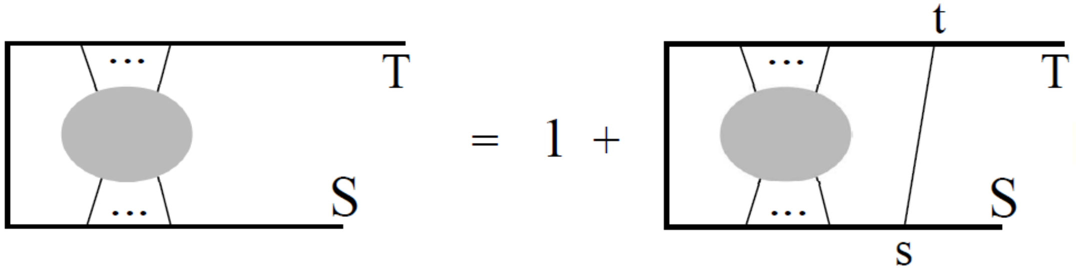
$$\langle W\rangle \approx 1-g^2N\int_0^T d\tau_1 d\tau_2\Big(\langle TA(\tau_1,0)A(\tau_2,L)\rangle+\Theta_1\cdot\Theta_2\langle T\Phi(\tau_1,0)\Phi(\tau_2,L)\rangle\Big)+\ldots$$

- Take $\lambda \to 0, \quad \Theta_1\cdot\Theta_2 \to \infty$ with the combination $\hat{\lambda} = \Theta_1\cdot\Theta_2\lambda$ fixed
- Selects so called planar ladder diagrams of scalar fields [Correa,Henn,Maldacena,Sever]



Summing the ladder diagrams

- The ladder diagrams can be resummed using the Bethe-Salpeter equation [Bethe,Salpeter][Erickson, Semenoff,Szabo, Zarembo][Erickson, Semenoff, Zarembo]



$$K(x_2;x_1) = \langle T\Phi(x_2)\Phi(x_1)\rangle$$

$$\Gamma(S,T) = 1 - \hat{\lambda}\int_0^T dt \int_0^S ds K(s,0;t,L)\Gamma(s,t)$$

$$\frac{\partial^2\Gamma(S,T)}{\partial S \partial T} = -\hat{\lambda}K(S,0;T,L)\Gamma(S,T)$$

$$x = S - T$$

$$\tau = S + T$$

$$(\partial_\tau^2 - \partial_x^2 + m_{eff}^2(x,\tau))\Gamma(x,\tau) = 0$$

$$m_{eff}^2 = \hat{\lambda}K$$

Vacuum ladders

- The vacuum scalar two point function is

$$K(\tau_2, L; \tau_1, 0) = \frac{1}{(\tau_1 - \tau_2)^2 + L^2}$$

- Separate variables

$$\Gamma(\tau, x) = e^{-\omega t}\psi(x)$$

- Schrödinger-like equation

$$\Big(- \partial_x^2 - \frac{\hat{\lambda}}{x^2 + L^2}\Big)\psi = \omega^2 \psi$$

- Analytic solution at strong coupling

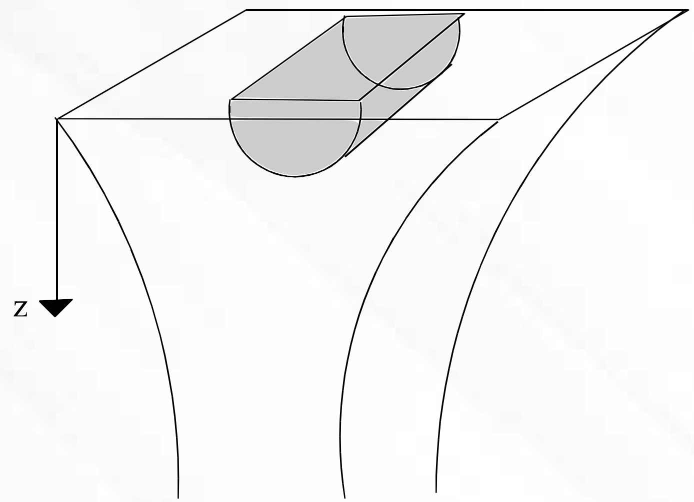
$$\langle W \rangle = \Gamma(0, 2T) \propto e^{T\frac{\sqrt{\hat{\lambda}}}{L}}$$

Wilson loops from AdS perspective

- A string endpoint on a brane acts as a charge in fundamental representation [Callan,Maldacena]
- Wilson loops are dual to macroscopic strings ending on the boundary on a loop [Maldacena]
- Semiclassically the Wilson loop is given by exponential of the on-shell string action

$$S_{NG} = \sqrt{\lambda} A$$

$$A = \int d^2\sigma \sqrt{|\det \partial_a X^\mu \partial_b X_\mu|}$$



(Euclidean) Vacuum comparison

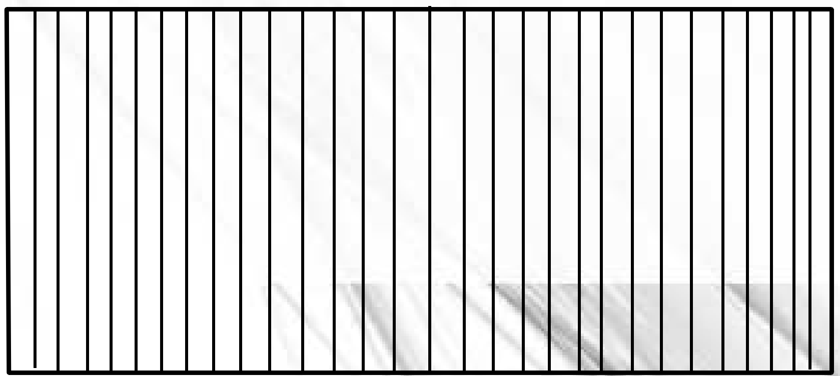
- A string hanging in AdS in semiclassical approximation [Maldacena]

$$A = T\left(\frac{1}{\epsilon} - \frac{1}{L}\right) \qquad S_{NG} = \sqrt{\lambda} A \qquad e^{-S_{NG}} \propto e^{T\frac{\sqrt{\lambda}}{L}}$$

- Agrees exactly with the ladder approximation result in the large T limit!
- The two methods give the same quark anti-quark potential
- Why?

Vacuum continued

- Weak coupling $$E = -\frac{\hat{\lambda}}{L}$$
- Strong coupling $$E = -\frac{\sqrt{\hat{\lambda}}}{L}$$
- L dependence follows from conformal symmetry
- At strong coupling the potential generated by highly virtual quanta separated by a time scale $\lambda^{-1/4}$ [Shuryak,Zahed]



Vacuum continued

- One can also compute energies of excited states of the flux tube using perturbation theory and string theory [Klebanov,Maldacena,Thorn]
- Weak coupling → No excited states with negative energy
- Strong coupling → Infinite number of excited states with negative energy
- Phase transition between the two behaviors as a function of the coupling
- Do the energies of excited states match? [work almost in progress… :)]

Real time Wilson loops

- How do the flux tubes react when we “kick” the system [VK]
- Eventually want to compare (qualitatively) with gravitational collapse calculations in AdS
- Take N=4 out of equilibrium by adding a time dependent mass term

$$S = -\frac{1}{g^2}\int d^4x \,\mathrm{Tr}\Big(\frac{1}{2}F^2 + (D\Phi^I)^2 + m^2(t)(\Phi^I)^2 - \left[\Phi^I, \Phi^J\right]^2 + \text{fermions}\Big) \qquad m^2(t) = \theta(-t)m_0^2$$

  High energy non-equilibrium state

- Take a limit $m_0 \gg 1/L$ to simplify calculations (“deep quench limit”)
- We need the scalar two point function for the Wilson loop
- Scalar two point function can be obtained through a Bogoliubov transformation [Calabrese,Cardy][Cardy,Sotiriadis]

Bogoliubov transformation

$t < 0$ $$\Phi(t,k) = \frac{g}{\sqrt{2\omega_0(k)}}\Big(Be^{-i\omega_0(k)t} + B^\dagger e^{i\omega_0(k)t}\Big)$$ $$\omega_0(k) = \sqrt{k^2 + m_0^2}$$

$t > 0$ $$\Phi(t,k) = \frac{g}{\sqrt{2\omega(k)}}\Big(Ae^{-i\omega(k)t} + A^\dagger e^{i\omega(k)t}\Big)$$ $$\omega(k) = |k|$$

$$A = \alpha B + \beta B^\dagger, \quad A^\dagger = \alpha^* B^\dagger + \beta^* B$$

$$\alpha = \frac{1}{2}\Big(\sqrt{\frac{\omega}{\omega_0}} + \sqrt{\frac{\omega_0}{\omega}}\Big), \quad \beta = \frac{1}{2}\Big(\sqrt{\frac{\omega}{\omega_0}} - \sqrt{\frac{\omega_0}{\omega}}\Big)$$ $$B|\psi\rangle = 0$$

$$\langle \Phi_a^I(t,x)\Phi_b^J(t',0)\rangle = g^2\delta_{ab}\delta^{IJ}\times,$$
$$\times \int \frac{d^3k}{(2\pi)^3}\frac{e^{ik\cdot x}}{2|k|}\Big(|\alpha|^2 e^{-i|k|(t-t')} + |\beta|^2 e^{i|k|(t-t')} + \alpha\beta e^{-i|k|(t+t')} + \alpha^*\beta^* e^{i|k|(t+t')}\Big)$$

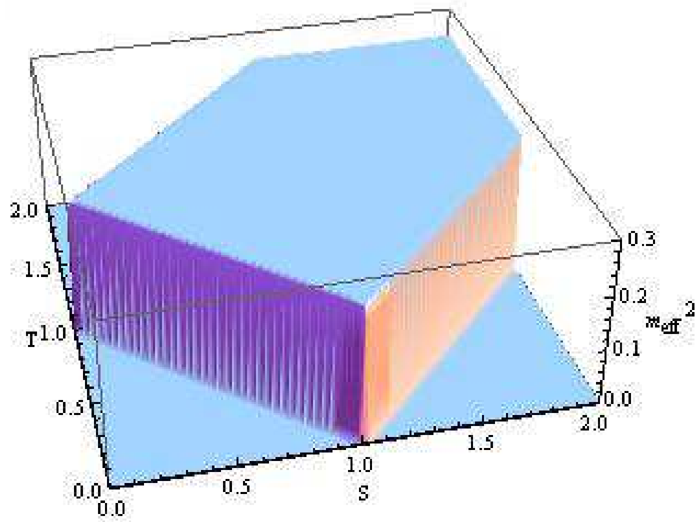
$$\langle \Phi(t,x)\Phi(t',0)\rangle \approx g^2 \int_0^\infty \frac{dk}{8\pi^2|x|}\sin(k|x|)\frac{m_0}{k}\Big(\cos k(t-t') - \cos k(t+t')\Big)$$

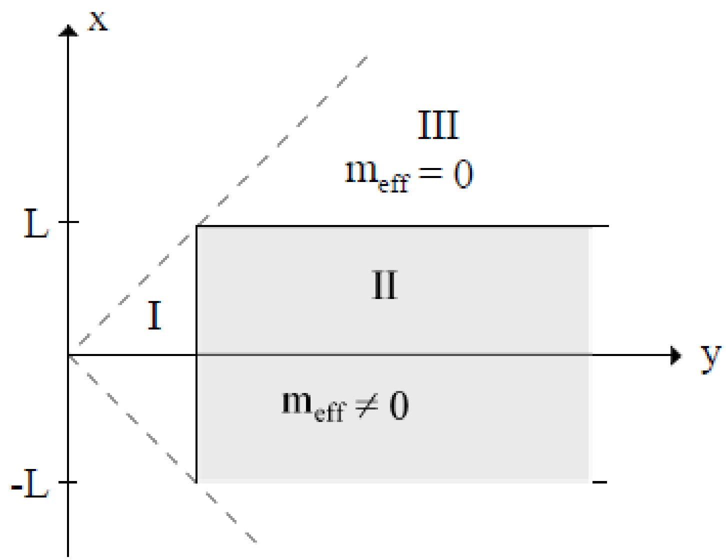
$$\langle \Phi(t,x)\Phi(t',0)\rangle \approx \frac{g^2 m_0}{16\pi}\frac{1}{|x|}\frac{1}{2}(\mathrm{sgn}(|x|+t-t') + \mathrm{sgn}(|x|-t+t') - \mathrm{sgn}(|x|+t+t') - \mathrm{sgn}(|x|-t-t'))$$

Real time Wilson loops

- With a time dependent two point function the mass term in the Bethe-Salpeter equation becomes time dependent

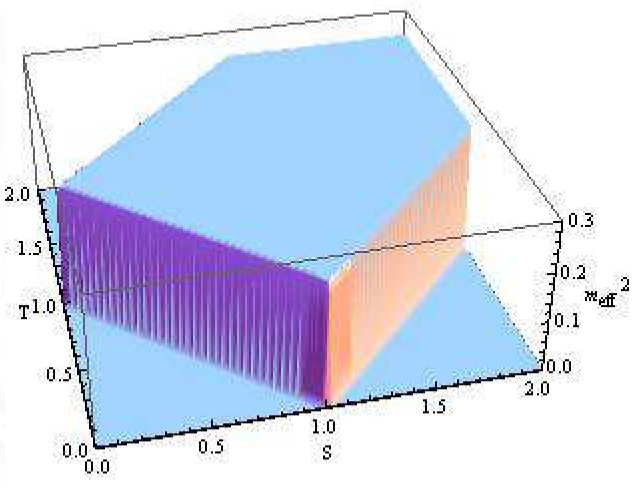
$$(\partial_y^2 - \partial_x^2 + m_{eff}^2)\Gamma(x, y) = 0$$





Real time Wilson loops

- The field leaks down the box potential

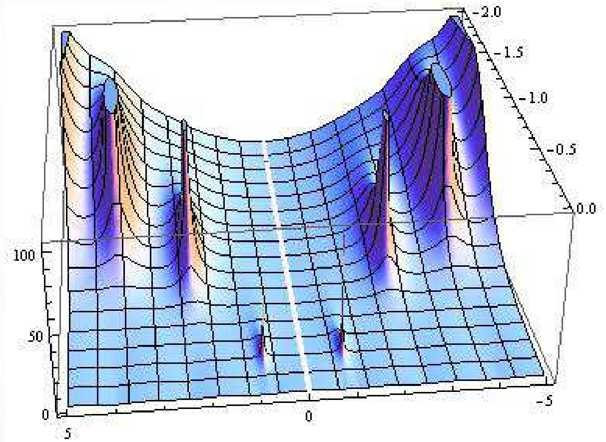


- Massive wave equation with leaking walls leads to exponential decay of $\Gamma(x,\tau)$ in "time"
- Value of the Wilson loop identified with the value of the field at the top of the box $\langle W \rangle = \Gamma(0,2T)$

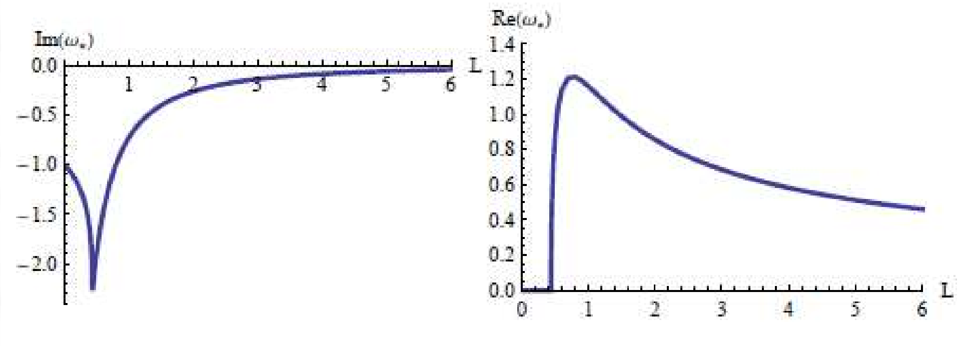
Real time Wilson loops

- Field equations exactly soluble outside the box
- Reduce the problem to massive KG equation on top of the box with a leaking boundary condition at the edge→ Quasinormal modes

$$\Gamma(x,\tau) = \int \frac{d\omega}{2\pi} e^{-i\omega\tau}\psi(x,\omega)$$



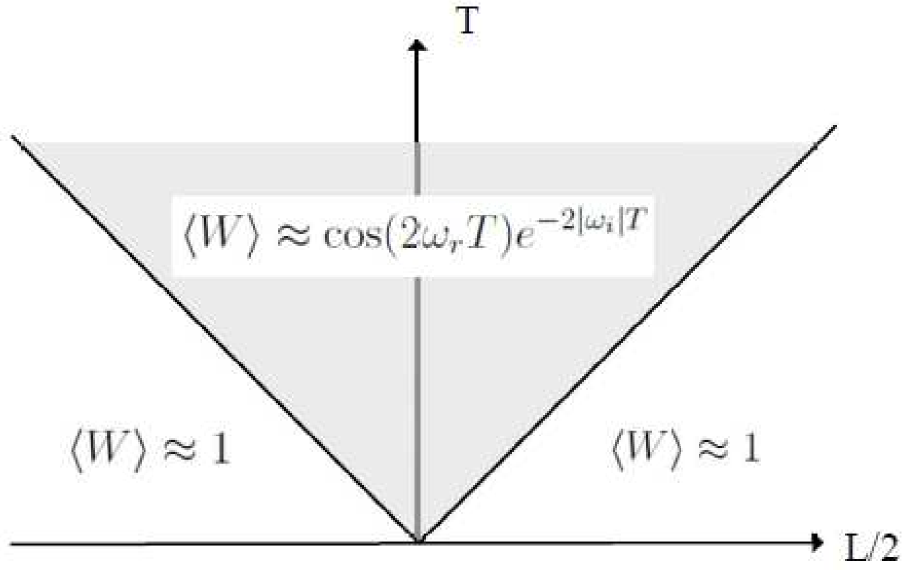
- The lowest mode



- Weak coupling: $\langle W(C)\rangle = e^{-\frac{\lambda m_0}{8\pi}\theta(T-\frac{L}{2})(T-\frac{L}{2})}$
- Strong coupling: $\langle W(C)\rangle \approx \cos(2\mu T)e^{-T\frac{8\pi^3}{\lambda m_0}\frac{1}{L^2}}$

Real time Wilson loops

- The grey part has the same quasinormal decay as in thermal equilibrium
- The longer the Wilson loop the slower it "thermalizes" with $t_{therm} \approx L/2$



- Thermalization can be made more precise by studying spatial Wilson loops (but this is also more boring) and one finds the same thermalization time

Non-equilibrium expectations from AdS

- Take the system out of equilibrium by injecting energy into the UV → Falls down in AdS and (usually) forms a black hole
  [Bhattacharyya,Minwalla][Wu]…
- Strings with small L thermalize first and the ones with large L slower with approximately $t_{therm} \approx L/2$ ✓
  [Balarubramanian,Bernamonti,de Boer,Copland,Craps,Keski-Vakkuri,Muller,Schafer,Shigemori,Staessens]
- Wilson loops in black hole background are expected to decay exponentially in the temporal length T (string falls towards the black hole exponentially fast and has to tunnel back up) ✓
  [Hayata,Nawa,Hatsuda]

Conclusions

- Bethe-Salpeter gives qualitatively reasonable and sometimes even quantitatively good answers that agree with strong coupling AdS calculations.
- We see the "top-down" thermalization from Bethe-Salpeter (this basically reflects causality)
- Also see quasinormal modes and a leaking boundary analogous to a black hole horizon (this sentence should not to be taken too seriously)

Thank you for listening!