# Conformal and disformal relations in extended models of gravity

Alexey Golovnev
*Saint Peterburg State University*

Helsinki
September 2016

The Einstein theory of gravity is extremely successful. And it has provided us with a breath-taking possibility to meaningfully discuss the origins of the whole Universe and its history.

**But:**

- Dark Matter is somewhat intriguing
- Dark Energy points at a technically very unnatural value of the cosmological constant
- We need a period of inflation in the very early Universe
- The underlying fundamental physics of inflation is not known
- Signatures of statistical anisotropy in the CMB radiation, and the lack of correlations at the largest angular scales

May be, we need to modify the theory of gravity?

In modified gravity, we often have two (or even more) different metrics.

For scalar-tensor models, the common relation between the two can be parametrised as follows

$$\hat{g}_{\mu\nu} = C(\phi, X) \cdot g_{\mu\nu} + D(\phi, X) \cdot (\partial_\mu \phi)(\partial_\nu \phi)$$

where $X \equiv g^{\mu\nu}(\partial_\mu \phi)(\partial_\nu \phi)$
with the conformal C and disformal D parts.

It is often linked to J. Bekenstein, gr-qc/9211017, even though he was occupied with determining the most general Finsler geometry one might afford for a viable theory of gravity.

In any case, those transformations are very important. For example, special (with no dependence on X) disformal transformations allow to do away with some non-minimal couplings in Horndeski models. And generalisation to general disformations leads to beyond Horndeski. (Zumalacarregui, Garcia-Bellido, 1308.4685.)

Let us consider a situation for which it is not obvious that disformations are relevant but still they are.

Let us consider a situation for which it is not obvious that disformations are relevant but still they are.

Mimetic Dark Matter

One of the biggest puzzles in modern cosmology is the nature of Dark Matter which persistently evades any kind of unequivocal detection outside the realm of gravitational interactions at galactic and cosmological scales. Not surprising, many attempts were made to model its observed effects by a suitable type of modified gravitational interaction, ranging from MOND to Hořava gravity.

Yet another, very interesting, model has been proposed by Mukhanov and Chamseddine, 1310.2790.

The idea is to represent the physical metric $g_{\mu\nu}$ in the Einstein-Hilbert variational principle as

$$g_{\mu\nu} = \tilde{g}_{\mu\nu}\tilde{g}^{\alpha\beta}(\partial_\alpha\phi)(\partial_\beta\phi)$$

with an auxiliary metric $\tilde{g}_{\mu\nu}$ and a scalar field $\phi$.
The conformal mode of $\tilde{g}$ is stripped off any physical significance, and its role is seemingly relegated to the scalar player.

However, as it follows from the definition, apart from the equation of motion

$$\bigtriangledown_\mu \left((G - T)\partial^\mu \phi\right) = 0,$$

the scalar field identically satisfies the constraint

$$g^{\alpha\beta}(\partial_\alpha \phi)(\partial_\beta \phi) = 1.$$

And after substituting it into the Einstein equations,

$$G_{\mu\nu} - T_{\mu\nu} - (G - T)(\partial_\mu \phi)(\partial_\nu \phi) = 0$$

with $G_{\mu\nu}$ and $T_{\mu\nu}$ being the Einstein tensor and the matter stress tensor respectively, and the trace quantites denoted by letters without indices, we see that the trace part of equations is satisfied identically.

Therefore, we have an extra freedom in the system. And effectively there is an extra contribution to the stress tensor of the form

$$\tilde{T}_{\mu\nu} = (G - T)(\partial_\mu \phi)(\partial_\nu \phi)$$

which amounts to introducing a pressureless dust with potential flow, or the *Mimetic Dark Matter*, with initial energy density being a mere integration constant.

It might be surprising that only by rearranging the parts of the metric, without introducing any new ingredients into the action, we end up with a new model at hand. Let us however look at this more attentively. Without loss of generality, assume that $\tilde{g}$ is unimodular. Therefore, the determinant of $g$ must be given by (the fourth power of) the factor of

$$\Omega(x) \equiv \tilde{g}^{\alpha\beta}(\partial_\alpha \phi)(\partial_\beta \phi).$$

To reproduce the standard Einstein equations, we need to vary the action

$$-\int d^4x R(g(\tilde{g}, \phi))$$

with respect to the metric $g$ including the $\Omega$ factor with the only restriction that the variation be vanishing at the boundary of the variation domain, most importantly at the initial and the final instants of time. This is not ensured by our definition of $\Omega$.

Actually, an equivalent formulation exists, AG 1310.2790.
As a first step, we introduce a set of Lagrange multipliers $\lambda^{\mu\nu}$ and write an action

$$S = -\int \left( R(g) + \lambda^{\mu\nu} \left( g_{\mu\nu} - \tilde{g}_{\mu\nu} \tilde{g}^{\alpha\beta} (\partial_\alpha \phi)(\partial_\beta \phi) \right) \right) \sqrt{-g} d^4 x.$$

The matter Lagrangian is totally omitted. However, if needed, its effect can be restored by simply subtracting $T_{\mu\nu}$ form $G_{\mu\nu}$ on all its occasions.

The Lagrange multiplier $\lambda$ is used to impose the normalisation condition for the scalar field.
Variation with respect to $\phi$ gives

$$\nabla_\mu (\lambda \partial^\mu \phi) = 0$$

where $\lambda \equiv \lambda^\mu_\mu \equiv g_{\mu\nu}\lambda^{\mu\nu}$.
The Einstein equations read

$$G_{\mu\nu} + \lambda_{\mu\nu} = 0,$$

and we see that the scalar equation of motion is reproduced.
Finally, we vary with respect to $\tilde{g}$ and get

$$\lambda^{\mu\nu}\tilde{g}^{\alpha\beta}(\partial_\alpha\phi)(\partial_\beta\phi) - \lambda^{\rho\sigma}\tilde{g}_{\rho\sigma}\tilde{g}^{\mu\alpha}(\partial_\alpha\phi)\tilde{g}^{\nu\beta}(\partial_\beta\phi) = 0.$$

which gives

$$\lambda_{\mu\nu} = \lambda(\partial_\mu\phi)(\partial_\nu\phi).$$

Actually, we can do better about the form of the action. We see that $\lambda_{\mu\nu}$ is fully determined by its trace. Therefore, it is tempting to leave only the trace part of the constraint-fixing term in the action,

$$S = -\int \left(R(g) + \lambda\left(1 - g^{\mu\nu}(\partial_\mu\phi)(\partial_\nu\phi)\right)\right)\sqrt{-g}\,d^4x.$$

We claim that this is an equivalent formulation of the model. Indeed, the Einstein equations take the form $G_{\mu\nu} + \lambda(\partial_\mu\phi)(\partial_\nu\phi) = 0$ as before. And the variation with respect to $\phi$ reproduces the scalar equation of motion.

It is remarkable that the conformal sector of general relativity can produce rather rich opportunities for theoretical model building if gently modified. Many years ago a unimodular gravity has been proposed to get rid off the problem of the cosmological constant. However, it was shown that the cosmological constant reappears as a constant of integration. Now we see that another dark sector – pressureless dust – can also appear via an integration constant in a model with modified variation in the conformal sector. It would be interesting to thoroughly explore possible extensions of mimetic Dark Matter model with potentially far-reaching phenomenological applications.

How is this related to disformal transformations?

## How is this related to disformal transformations?

The answer is in N. Deruelle, J. Rua, 1407.0825:

*"We show that Einstein's equations for gravity are generically invariant under 'disformations'. We also show that the particular subclass when this is not true yields the equations of motion of 'Mimetic Gravity'."*

Let's try to redo the Mimetic Dark Matter proposal with the following definition of the physical metric:

$$g_{\mu\nu} = C(P) \cdot \tilde{g}_{\mu\nu} + D(P) \cdot (\partial_\mu \phi)(\partial_\nu \phi)$$

where

$$P \equiv \tilde{g}^{\alpha\beta}(\partial_\alpha \phi)(\partial_\beta \phi).$$

The action is taken to be

$$\int \left( R(g) + \lambda^{\mu\nu} \left( g_{\mu\nu} - C(P)\tilde{g}_{\mu\nu} - D(P) \cdot (\partial_\mu \phi)(\partial_\nu \phi) \right) \right) \sqrt{-g}\, d^4 x.$$

The standard mimetic case is $C(P) = P$ and $D = 0$.

Variation with respect to $\phi$ gives

$$\nabla_\mu \left( \left( D\lambda^{\mu\nu} + \left( C' \tilde{g}_{\alpha\beta} + D'(\partial_\alpha \phi)(\partial_\beta \phi) \right) \lambda^{\alpha\beta} \tilde{g}^{\mu\nu} \right) \partial_\nu \phi \right) = 0.$$

The Einstein equations read $G^{\mu\nu} = \lambda^{\mu\nu}$.

Finally, we vary the action with respect to $\tilde{g}$ and get

$$\lambda^{\mu\nu} = \lambda^{\rho\sigma} \left( \frac{C'}{C} \tilde{g}_{\rho\sigma} + \frac{D'}{C} (\partial_\rho \phi)(\partial_\sigma \phi) \right) \tilde{g}^{\mu\alpha} \tilde{g}^{\nu\beta} (\partial_\alpha \phi)(\partial_\beta \phi).$$

Let us define a scalar quantity

$$\lambda \equiv \lambda^{\rho\sigma} \left( \frac{C'}{C} \tilde{g}_{\rho\sigma} + \frac{D'}{C} (\partial_\rho \phi)(\partial_\sigma \phi) \right)$$

which gives:

$$\lambda^{\mu\nu} = \lambda \tilde{g}^{\mu\alpha} \tilde{g}^{\nu\beta} (\partial_\alpha \phi)(\partial_\beta \phi).$$

Lowering the indices by the physical metric, $\lambda_{\mu\nu} \equiv g_{\mu\alpha} g_{\nu\beta} \lambda^{\alpha\beta}$, we get

$$\lambda_{\mu\nu} = \lambda (C + DP)^2 (\partial_\mu \phi)(\partial_\nu \phi).$$

To understand the structure of equations better, we multiply

$$\lambda^{\mu\nu} = \lambda \tilde{g}^{\mu\alpha} \tilde{g}^{\nu\beta} (\partial_\alpha \phi)(\partial_\beta \phi)$$

by a factor of

$$\frac{C'}{C} \tilde{g}_{\rho\sigma} + \frac{D'}{C} (\partial_\rho \phi)(\partial_\sigma \phi)$$

and get

$$\lambda = \lambda \left( \frac{C'}{C} P + \frac{D'}{C} P^2 \right).$$

Generically, it means that $\lambda_{\mu\nu} = 0$ and the standard GR is reproduced.

However, in a special case of (including the standard mimetic)

$$C'P + D'P^2 = C$$

it gives no constraints, and the $\lambda$ is a new integration constant. And we have again an effective pressureless dust with

$$(\partial\phi)^2 = \frac{P}{C + DP}.$$

Let us now include curvature-dependence into the coefficients...

...and obtain nice relations to non-local models
AG, Koivisto, Sandstad, 1509.06552

Amendola, Enqvist and Koivisto proposed to consider a funny type of non-metricity:

$$\hat{g}_{\mu\nu} = C(\mathcal{R}) \cdot g_{\mu\nu}$$

where $\mathcal{R} \equiv g^{\mu\nu} \hat{R}_{\mu\nu}$ – the so-called C-models.

One can also consider the more complicated arguments for the C-function.

It can also be generalised to D-models:

$$\hat{g}_{\mu\nu} = C(\mathcal{R}) \cdot g_{\mu\nu} + D(\mathcal{R}) \cdot \hat{R}_{\mu\nu}.$$

We consider the action

$$S = \int d^n x \sqrt{-g} g^{\mu\nu} \hat{R}_{\mu\nu}$$

where

$$\hat{g}_{\mu\nu} = C(\mathbf{R}) g_{\mu\nu} + D(\mathbf{R}) \hat{R}_{\mu\nu}$$

where the functions $C$ and $D$ depend on the matrix $\mathbf{R}^{\mu}_{\nu} \equiv g^{\mu\alpha} \hat{R}_{\alpha\nu}$ via its scalar invariants such as $\mathrm{Tr}\mathbf{R} = g^{\mu\nu} \hat{R}_{\mu\nu} \equiv \mathcal{R}$, $\mathrm{Tr}\mathbf{R}^2$ et cetera.

We would like to study the perturbations around the double Minkowski solution, and denote at $\hat{R}_{\mu\nu} \to 0$

$$C(\mathbf{R}) = 1 + c_{,1} \mathcal{R} + ....$$

and

$$D(\mathbf{R}) = d_0 + ....$$

We have for first-order fluctuations ($g \equiv \eta + h$):

$$\hat{h}_{\mu\nu} = h_{\mu\nu} + c_{,1}\mathcal{R}\eta_{\mu\nu} + d_0\hat{R}_{\mu\nu}$$

and for curvatures:

$$\hat{R}_{\mu\nu} = \frac{1}{2}\left(\partial^2_{\mu\alpha}\hat{h}^{\alpha}_{\nu} + \partial^2_{\nu\alpha}\hat{h}^{\alpha}_{\mu} - \Box\hat{h}_{\mu\nu} - \partial^2_{\mu\nu}\hat{h}^{\alpha}_{\alpha}\right) + \ldots.$$

and

$$\mathcal{R} = \partial^2_{\mu\nu}\hat{h}^{\mu\nu} - \Box\hat{h}^{\mu}_{\mu} + \ldots$$

where the indices are raised with $\eta_{\mu\nu}$.

The second order action reads

$$S = \int d^n x \left( \eta^{\mu\nu} \delta^{(2)} \hat{R}_{\mu\nu} + \delta^{(1)} \left( \sqrt{-g} g^{\mu\nu} \right) \cdot \delta^{(1)} \hat{R}_{\mu\nu} \right).$$

Using

$$\delta^{(1)} \left( \sqrt{-g} g^{\mu\nu} \right) = -h^{\mu\nu} + \frac{1}{2} \eta^{\mu\nu} h^{\alpha}_{\alpha}$$

and taking into account that

$$\eta^{\mu\nu} \delta^{(2)} \hat{R}_{\mu\nu} + \delta^{(1)} \left( \sqrt{-\hat{g}} \hat{g}^{\mu\nu} \right) \cdot \delta^{(1)} \hat{R}_{\mu\nu} = \delta^{(2)} \left( \sqrt{-\hat{g}} \hat{R} \right)$$

we see that the model is equivalent at quadratic level to

$$S = \int d^n x \sqrt{-\hat{g}} \left( \hat{R} - \frac{(n-2)c_{,1} + d_0}{2} \hat{R}^2 + d_0 \hat{R}^{\mu\nu} \hat{R}_{\mu\nu} \right)$$

and therefore contains ghosts unless in a pure C-theory.

For a non-perturbative treatment, let us also define a matrix $\hat{\mathbf{R}}^{\mu}_{\nu} \equiv \hat{g}^{\mu\alpha}\hat{R}_{\alpha\nu}$. We have then:

$$g_{\mu\nu} = \frac{\hat{g}_{\mu\alpha}}{C}\left(\delta^{\alpha}_{\nu} - D\hat{R}^{\alpha}_{\nu}\right)$$

or, in matrix notation,

$$g = \frac{1}{C} \cdot \hat{g}\left(I - D\hat{\mathbf{R}}\right)$$

and

$$g^{-1} = C \cdot \left(I - D\hat{\mathbf{R}}\right)^{-1} \hat{g}^{-1}.$$

And then the action reads

$$S = \int d^n x \sqrt{-\det \hat{g}} \cdot \frac{\sqrt{\det\left(\mathrm{I} - \mathrm{D}\hat{\mathbf{R}}\right)}}{C^{\frac{n-2}{2}}} \cdot \mathrm{Tr}\left(\left(\mathrm{I} - \mathrm{D}\hat{\mathbf{R}}\right)^{-1} \hat{\mathbf{R}}\right).$$

In particular, for C-models one generically gets $f(R)$-type models.

Of course, one has to rephrase the arguments of $C$ and $D$ functions in terms of hatted quantities. For that we multiply the equation for $g^{-1}$ by $\hat{R}_{\mu\nu}$ and get algebraic equation for $\mathbf{R}$ in terms of $\hat{\mathbf{R}}$:

$$\mathbf{R} = C(\mathbf{R}) \cdot \left(I - D(\mathbf{R}) \cdot \hat{\mathbf{R}}\right)^{-1} \hat{\mathbf{R}}.$$

Above we eliminated the physical metric $g_{\mu\nu}$ and wrote the resulting (quadratic) action in terms of the other metric. This turns out to be easier in practice than eliminating $\hat{g}$ in order to get a (possibly non-local) action for $g$. However, the latter can also be done at least for C-theories at perturbative level around double Minkowski.

Let us first naively start with the linear level relation

$$\hat{h}_{\mu\nu} = h_{\mu\nu} + c_{,1}\mathcal{R}\eta_{\mu\nu}$$

between the metrics.
If we write for convenience the general Weyl relation as

$$\hat{g}_{\mu\nu} = e^{2\rho} g_{\mu\nu},$$

then

$$\mathcal{R} = R - (n-1)(n-2)(\partial\rho)^2 - 2(n-1)\Box\rho$$

.

It is not difficult to find the $\rho$-factor to the first order in perturbations. To this end, we first write

$$\hat{h}_{\mu\nu} = h_{\mu\nu} + c_{,1}\eta_{\mu\nu}\left(\partial^2_{\alpha\beta}\hat{h}^{\alpha\beta} - \Box\hat{h}^{\alpha}_{\alpha}\right) ,$$

and then note that only the trace part is changed,

$$\hat{h}_{\mu\nu} = h_{\mu\nu} + \frac{1}{n}\eta_{\mu\nu}\left(\hat{h}^{\alpha}_{\alpha} - h^{\alpha}_{\alpha}\right) ,$$

and that it thus can be solved as

$$\hat{h}^{\mu}_{\mu} = \frac{1}{1+(n-1)c_{,1}\Box}\left(h^{\mu}_{\mu} + nc_{,1}\partial^2_{\mu\nu}h^{\mu\nu} - c_{,1}\Box h^{\alpha}_{\alpha}\right) ,$$

which with this accuracy is equivalent to Weyl transformation with

$$\rho = \frac{1}{2n}(\hat{h}^{\alpha}_{\alpha} - h^{\alpha}_{\alpha}) = \frac{c_{,1}\left(\partial^2_{\alpha\beta}h^{\alpha\beta} - \Box h^{\alpha}_{\alpha}\right)}{2(1+(n-1)c_{,1}\Box)} = \frac{c_{,1}}{2(1+(n-1)c_{,1}\Box)}R\,.$$

Therefore the action is

$$S = \int \mathrm{d}^n x \sqrt{-\det(\mathrm{g})} \left( R - (n-1)(n-2)(\partial\rho)^2 - 2(n-1)\Box\rho \right) ,$$

and, if the surface terms can be omitted (otherwise we must use the C-relation to the second order including the $c_{,2}$ part), we get

$$S = \int \mathrm{d}^n x \frac{\sqrt{-\det(\mathrm{g})}}{1+(n-1)c_{,1}\Box} \left( R + R\frac{(n-1)(n-2)c_{,1}^2\Box}{4(1+(n-1)c_{,1}\Box)^2} R \right)$$

$$= \int \mathrm{d}^n x \sqrt{-\det(\mathrm{g})} \left( R + R\frac{(n-1)(n-2)c_{,1}^2\Box}{4(1+(n-1)c_{,1}\Box)^2} R \right)$$

One has to be careful about the class of variations and boundary terms especially when the coefficients $c_{,i}$ are promoted into (inverse) derivative operators. For example, if we take a non-local function with $c_{,1} \sim \frac{1}{\Box}$, then omitting the $\Box\rho$-term does not seem even naively appropriate. In this case one might argue that we need to know $\rho$ to the second order in perturbations including the knowledge of $c_{,2}$.

The action contains the linear (surface) term $\partial^2_{\mu\nu}\hat{h}^{\mu\nu} - \Box\hat{h}^\mu_\mu$ which requires the second order accuracy in the relation between the metrics.However, an accurate quadratic level treatment of our action is of course possible. We have

$$g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu} + h^{\mu\alpha}h^\nu_\alpha + \mathcal{O}\left(h^3\right) ,$$

the metric determinant

$$\sqrt{-\det(g)} = 1 + \frac{1}{2}h^\mu_\mu - \frac{1}{4}h_{\mu\nu}h^{\mu\nu} + \frac{1}{8}\left(h^\mu_\mu\right)^2 + \mathcal{O}\left(h^3\right),$$

connection coefficients

$$\hat{\Gamma}^\alpha_{\mu\nu} = \frac{1}{2}\left(\partial_\mu\hat{h}^\alpha_\nu + \partial_\nu\hat{h}^\alpha_\mu - \partial^\alpha\hat{h}_{\mu\nu}\right) - \frac{1}{2}\hat{h}^{\alpha\beta}\left(\partial_\mu\hat{h}_{\beta\nu} + \partial_\nu\hat{h}_{\beta\mu} - \partial_\beta\hat{h}_{\mu\nu}\right) + \ldots$$

and then oine can have (somewhat cumbersome) explicit expressions for the curvatures and Lagragian density.

Now we see that, indeed, in order to go to the $\hat{g}$ picture we need the relation between $h$ and $\hat{h}$ only to the first order since $h$ (which we want to exclude) enters only in quadratic terms. However, transition to the picture of $g$ requires the second order accuracy for $\hat{h}$ in terms of $h$ if we are to keep proper track of the surface term $\partial^2_{\mu\nu}\hat{h}^{\mu\nu} - \Box\hat{h}^{\mu}_{\mu}$ in the action. Then the needed accuracy is

$$\hat{h}_{\mu\nu} = \left(1 + c_{,1}\delta\mathcal{R}^{(1)}\right) h_{\mu\nu} + \left(c_{,1}\delta\mathcal{R}^{(2)} + \frac{1}{2}c_{,2}\left(\delta\mathcal{R}^{(1)}\right)^2\right)\eta_{\mu\nu}.$$

If we naively drop the surface terms, especially with non-local coefficients, then there is an apparent discrepancy between the two pictures. A suitable special case is given in $n = 4$ by choosing $c_{,1} = \frac{2}{3\Box}$ and $D = 0$ for simplicity. We see that $\frac{1}{1+(n-1)\Box c_{,1}} = \frac{1}{3}$ and the $g$-picture action reduces to

$$S \approx \int \mathrm{d}^4x\sqrt{-\det(\mathrm{g})}\left(\frac{1}{3}R + \frac{2}{81}R\frac{1}{\Box}R\right),$$

which resembles the structure in the $\hat{g}$ picture that now reduces to

$$S \approx \int \mathrm{d}^nx\sqrt{-\det(\hat{\mathrm{g}})}\left(\hat{R} - \frac{2}{3}\hat{R}\frac{1}{\Box}\hat{R}\right).$$

however the coefficients and the physical spectra are different.

Nevertheless, now we can explicitly check that, modulo the integration by parts (and higher order terms), the actions are equal. Indeed, with $\mathcal{R}=\frac{1}{3}R+\frac{2}{81}R\frac{1}{\Box}R$ we have

$$\hat{R}=\frac{\mathcal{R}}{C(\mathcal{R})}=\mathcal{R}-\mathcal{R}c_{,1}\mathcal{R}=\frac{1}{3}R-\frac{4}{81}R\frac{1}{\Box}R\,,$$

and then $\sqrt{-\hat{g}}=C^2(\mathcal{R})\sqrt{-g}=\left(1+\frac{4}{9\Box}R\right)\sqrt{-g}$ to the linear order, and for the hatted action we obtain

$$\sqrt{-\mathrm{det}(\hat{\mathrm{g}})}\left(\hat{R}-\frac{2}{3}\hat{R}\frac{1}{\Box}\hat{R}\right)=\sqrt{-\mathrm{det}(\mathrm{g})}\left(\frac{1}{3}R+\frac{2}{81}R\frac{1}{\Box}R\right)\,,$$

which proves the mathematical equality. However, even the signs of the correction terms differ in the two pictures, and it appears nontrivial that they ought to describe the same physics.

This can be clarified by checking the correspondence between different frames at the level of equations of motion, as we shall do in the following without fixing any of the coefficients $c_{,i}$.

One can easily find the equations of motion for the action

$$S = \int d^n x \sqrt{-\hat{g}} \left( \hat{R} - \frac{(n-2)c_{,1}}{2} \hat{R}^2 \right)$$

to be

$$\left(1 - (n-2)c_{,1} \hat{R}\right) \hat{R}_{\mu\nu} + (n-2)c_{,1} \left( \hat{\bigtriangledown}_\mu \hat{\bigtriangledown}_\nu - \hat{g}_{\mu\nu} \hat{\Box} \right) \hat{R} \\ - \frac{1}{2} \left( \hat{R} - \frac{(n-2)c_{,1}}{2} \hat{R}^2 \right) \hat{g}_{\mu\nu} = 0\,,$$

which at the linear level boil down to

$$\hat{R}_{\mu\nu} - \frac{1}{2}\hat{R}\eta_{\mu\nu} + (n-2)c_{,1}\left(\partial_\mu\partial_\nu - \eta_{\mu\nu}\Box\right)\hat{R} = 0\,,$$

or, in explicit metric variables:

$$\partial^2_{\mu\alpha}\hat{h}^\alpha_\nu + \partial^2_{\nu\alpha}\hat{h}^\alpha_\mu - \Box\hat{h}_{\mu\nu} - \partial^2_{\mu\nu}\hat{h}^\alpha_\alpha$$
$$+\left(2(n-2)c_{,1}\left(\partial^2_{\mu\nu} - \eta_{\mu\nu}\Box\right) - \eta_{\mu\nu}\right)\cdot\left(\partial^2_{\alpha\beta}\hat{h}^{\alpha\beta} - \Box\hat{h}^\alpha_\alpha\right) = 0\,.$$

Let us choose the harmonic gauge for $\hat{h}$:

$$\partial_\mu \hat{h}^{\mu\nu} = \frac{1}{2}\partial^\nu \hat{h}^\mu_\mu\,.$$

Then it is easy to see that

$$\hat{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}c_{,1}\eta_{\mu\nu}\Box\hat{h}^\alpha_\alpha\,,$$

or

$$\hat{h}^\mu_\mu = \frac{1}{1+\frac{n}{2}c_{,1}\Box}\cdot h^\mu_\mu\,,$$

and, in the picture without the hats, we have the gauge

$$\partial_\mu h^{\mu\nu} = \frac{1}{2}\partial^\nu \frac{1+c_{,1}\Box}{1+\frac{n}{2}c_{,1}\Box}h^\mu_\mu\,.$$

Let us now apply the harmonic gauge to the field equation:

$$\Box \hat{h}_{\mu\nu} + \frac{1}{2}\left(2(n-2)c_{,1}\left(\partial^2_{\mu\nu} - \eta_{\mu\nu}\Box\right) - \eta_{\mu\nu}\right)\cdot \Box \hat{h}^{\alpha}_{\alpha} = 0\,.$$

The traceless part obeys the wave equation with a source term dependent on the trace part, and for the trace part we have:

$$(n-2)\left(1 + 2(n-1)c_{,1}\Box\right)\cdot \Box \hat{h}^{\mu}_{\mu} = 0\,,$$

which is equivalent to

$$(n-2)\frac{1+2(n-1)c_{,1}\Box}{1+\frac{n}{2}c_{,1}\Box}\cdot \Box h^{\mu}_{\mu} = 0\,.$$

$$(n-2)\frac{1+2(n-1)c_{,1}\Box}{1+\frac{n}{2}c_{,1}\Box}\cdot\Box h^{\mu}_{\mu}=0$$

The question is whether we can get it also directly from the non-local action?

We find that the equation of motion to first order resulting from the action

$$S = \int \mathrm{d}^n x \sqrt{-\det(\mathrm{g})} \left( R + R \frac{(n-1)(n-2)c_{,1}^2 \Box}{4(1+(n-1)c_{,1}\Box)^2} R \right)$$

is

$$R_{\mu\nu} - \frac{1}{2} R \eta_{\mu\nu} - \frac{(n-1)(n-2)c_{,1}^2 \Box}{2\left(1+(n-1)c_{,1}\Box\right)^2} \left(\partial_\mu \partial_\nu - \eta_{\mu\nu}\Box\right) R = 0$$

which in metric variables takes the form of

$$\partial^2_{\mu\alpha} h^\alpha_\nu + \partial^2_{\nu\alpha} h^\alpha_\mu - \Box h_{\mu\nu} - \partial^2_{\mu\nu} h^\alpha_\alpha$$
$$- \left[ \eta_{\mu\nu} + \frac{(n-1)(n-2)c_{,1}^2 \Box}{\left(1+(n-1)c_{,1}\Box\right)^2} \left(\partial_\mu \partial_\nu - \eta_{\mu\nu}\Box\right) \right] \left( \partial_{\alpha\beta} h^{\alpha\beta} - \Box h^\alpha_\alpha \right) = 0.$$

Employing the chosen gauge

$$\partial_\mu h^{\mu\nu} = \frac{1}{2}\partial^\nu \frac{1 + c_{,1}\Box}{1 + \frac{n}{2}c_{,1}\Box} h^\mu_\mu$$

this equation becomes:

$$0 = \partial^2_{\mu\nu} \frac{1 + c_{,1}\Box}{1 + \frac{n}{2}c_{,1}\Box} h^\alpha_\alpha - \Box h_{\mu\nu} - \partial^2_{\mu\nu} h^\alpha_\alpha$$
$$+ \left[\eta_{\mu\nu} + \frac{(n-1)(n-2)c^2_{,1}\Box}{(1+(n-1)c_{,1}\Box)^2}(\partial_\mu\partial_\nu - \eta_{\mu\nu}\Box)\right] \frac{1+(n-1)c_{,1}\Box}{2+nc_{,1}\Box}\Box h^\alpha_\alpha$$

Again we study the trace equation and get:

$$(n-2)\frac{1 + 2(n-1)c_{,1}\Box}{(2+nc_{,1}\Box)\cdot(1+(n-1)c_{,1}\Box)} \cdot \Box h^\mu_\mu = 0$$

The difference between

$$(n-2)\frac{1+2(n-1)c_{,1}\Box}{1+\frac{n}{2}c_{,1}\Box}\cdot\Box h^{\mu}_{\mu}=0$$

and

$$(n-2)\frac{1+2(n-1)c_{,1}\Box}{(2+nc_{,1}\Box)\cdot(1+(n-1)c_{,1}\Box)}\cdot\Box h^{\mu}_{\mu}=0$$

is in the nonlocal operator $\frac{1}{2(1+(n-1)c_{,1}\Box)}$ which is the price for making a change of variables $\hat{h}\to h$ with derivatives.
This is always the case, of couse.

Note that varying directly the action

$$S = \int \mathrm{d}^n x \frac{\sqrt{-\mathrm{det(g)}}}{1+(n-1)c_{,1}\Box} \left( R + R\frac{(n-1)(n-2)c_{,1}^2\Box}{4(1+(n-1)c_{,1}\Box)^2} R \right)$$

we would get yet another power of the non-local factor:

$$(n-2)\frac{1+2(n-1)c_{,1}\Box}{(2+nc_{,1}\Box)\cdot(1+(n-1)c_{,1}\Box)^2}\cdot\Box h^{\mu}_{\mu} = 0$$

due to the overall factor of $\frac{1}{(1+(n-1)c_{,1}\Box)}$ in the action. Once more, we see that it is very important to consistently treat the classes of variations and surface terms when dealing with such models.

And the case with $c_{,i} \propto \frac{1}{\Box}$ is even more delicate since the linear term in the curvature scalar is no longer a surface one.

Let us illustrate the subtleties by a toy model with two scalar fields $\phi(x)$ and $\psi(x)$

$$S = \int \mathrm{d}^n x \cdot (1 + \psi(x)) \left(\Box\phi(x) - (\partial\phi(x))^2\right)$$

constrained by relation

$$\phi = \psi + c_{,1} \left(\Box\phi - (\partial\phi)^2\right) + c_{,1}\psi\Box\phi + c_{,2}(\Box\phi)^2 + \dots$$

which generalises the C-model relation for the two metrics being $1 + \phi$ and $1 + \psi$.

For the quadratic action in the $\phi$-picture, it is enough to solve for $\psi$ up to the linear order. We substitute

$$\psi = \phi - c_{,1}\Box\phi + \dots$$

and get

$$S = -\int \mathrm{d}^n x \cdot \left(2(\partial\phi)^2 + c_{,1}(\Box\phi)^2\right)$$

which yields the equation of motion

$$2\Box\phi - c_{,1}\Box^2\phi = 0$$

with higher order derivatives stemming from the derivative relation between the fields.

If we are not allowed to throw away the surface tems, then the opposite transition requires solving for $\phi$ to second order:

$$\phi = \frac{1}{1-c_{,1}\Box}\left(\psi + c_{,1}\left(\psi\frac{\Box}{1-c_{,1}\Box}\psi + \left(\partial\frac{\psi}{1-c_{,1}\Box}\right)^2\right) + c_{,2}\left(\frac{\Box}{1-c_{,1}\Box}\psi\right)^2 + \dots\right).$$

The second order action is

$$S = \int d^4x\left(\psi\frac{\Box}{1-c_{,1}\Box}\psi - \left(\partial\frac{\psi}{1-c_{,1}\Box}\right)^2 + \frac{\Box}{1-c_{,1}\Box}\left(\psi + c_{,1}\left(\psi\frac{\Box}{1-c_{,1}\Box}\psi + \left(\partial\frac{\psi}{1-c_{,1}\Box}\right)^2\right) + c_{,2}\left(\frac{\Box}{1-c_{,1}\Box}\psi\right)^2\right)\right).$$

Omitting the surface terms, we get the equation of motion

$$\left(\frac{\Box}{(1 - c_{,1}\Box)^2} + \frac{\Box}{1 - c_{,1}\Box}\right)\psi = 0$$

which easily transforms to

$$\frac{1}{1 - c_{,1}\Box}(2\Box\phi - c_{,1}\Box^2)\phi = 0$$

The difference between the two pictures amounts to the non-local operator which has been used to go from $\psi$ to $\phi$.

Again, if $c_{,i} \propto \frac{1}{\Box}$, the difference is much more profound, at least with the standard choice of boundary conditions. Indeed, in this case the term with $c_{,2}$ in the $\psi$-action is obviously not the surface one, and therefore the equations of motion differ by an extra non-trivial term.

Equivalence between different pictures is a tricky issue that requires exquisite care.

We have nevertheless established that under reasonable assumptions, we can effectively regard wide classes of metric-affine gravities as nonlocal metric theories.

And there is still definitely much more of what we can do with transformations of conformal and disformal types!

Thank you for your attention!