THE SCALE INVARIANT INFLATIONARY UNIVERSE

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OUTLINE

- Introduction: why f(R)?
- Reconstructing the inflationary f(R) theory
- Breaking of scale-invariance with 1-loop corrections
- Spontaneous and classical breaking of scale-invariance
- Road ahead
- Conclusions

Intro I: F(R) Gravity

Einstein-Hilbert Action: $$S = \frac{1}{8\pi G}\int d^4x\sqrt{g}(R-2\Lambda)+S_m$$

Quadratic actions like $$\mathcal{L} = \frac{R}{8\pi G}+\alpha R^2+\beta R_{\mu\nu}R^{\mu\nu}+\mathcal{L}_m$$

are motivated by renormalizability. So why not $\mathcal{L}\sim f(R)$?

Not always equivalent to scalar-tensor gravity: example

$$\boxed{\mathcal{L}_J = \sqrt{g}R^2} \qquad \tilde{g}_{\mu\nu} = \Omega^2 g_{\mu\nu} \qquad \boxed{\Omega^2 = \frac{4R}{M^2}}$$

Weyl rescaling to the Einstein frame:

$$\mathcal{L}_E = \sqrt{\tilde{g}}\tilde{R}-\frac{1}{2}(\tilde{\partial}\psi)^2-\frac{M^4}{16} \qquad \psi = 2\sqrt{6}M\ln\Omega$$

In Jordan frame we have the solution (for flat RW metric)

$$H(N) = \left(c_1 + c_2\, e^{-3N}\right)^{2/3} \qquad\qquad N = \ln a$$

Interpolates between a de Sitter space ($c_2 = 0$) with arbitrary cosmological constant and a radiation-dominated Universe ($c_1 = 0$) with R = 0 i.e.

$$\boxed{\Omega = 0}$$

thus the Weyl transformations not well-defined.

Except for these cases, f(R) gravity is classically equivalent to Brans-Dicke theory with $\omega_{BD} = 0$. (see De Felice's review). No definitive proof yet that the equivalence between J and E frame extends at the quantum level.

Intro II: scale invariance

Cosmological observations tells us that $1 - n_s \ll 0$,
so the spectrum of curvature perturbation is
quasi scale-invariant, and that $r \ll 1$

This can be easily obtained in the Starobinski model

$$\frac{\mathcal{L}}{\sqrt{g}} = f(R) = \frac{M_p^2 R}{2} + \frac{R^2}{36M^2} \qquad M_p/M \sim 10^5$$

The pure quadratic term is renormalizable and ghost-free.

Strumia et al.: Nature does not contain any scale; mass scales are generated by quantum effects.
Gravitational ghosts are present [JHEP06(2014)080].

What is the “best” inflationary f(R) in vacuum?

[MR et al. JCAP08(2014)015]

The equations of motion for on a flat RW spacetime are:

$$\boxed{S = \int d^4x \sqrt{g} f(R)}$$

$$3FH^2 = \frac{1}{2}(FR - f) - 3H\dot{F}, \quad \ddot{F} = -2F\dot{H} + H\dot{F} \qquad F(R) = \frac{df(R)}{dR}$$

$$\boxed{f(R) = f_0 R^\zeta}, \qquad \boxed{\zeta = \frac{4 - \alpha - \alpha'}{2(1-\alpha) - \alpha'}} \qquad \alpha(N) = F'/F$$

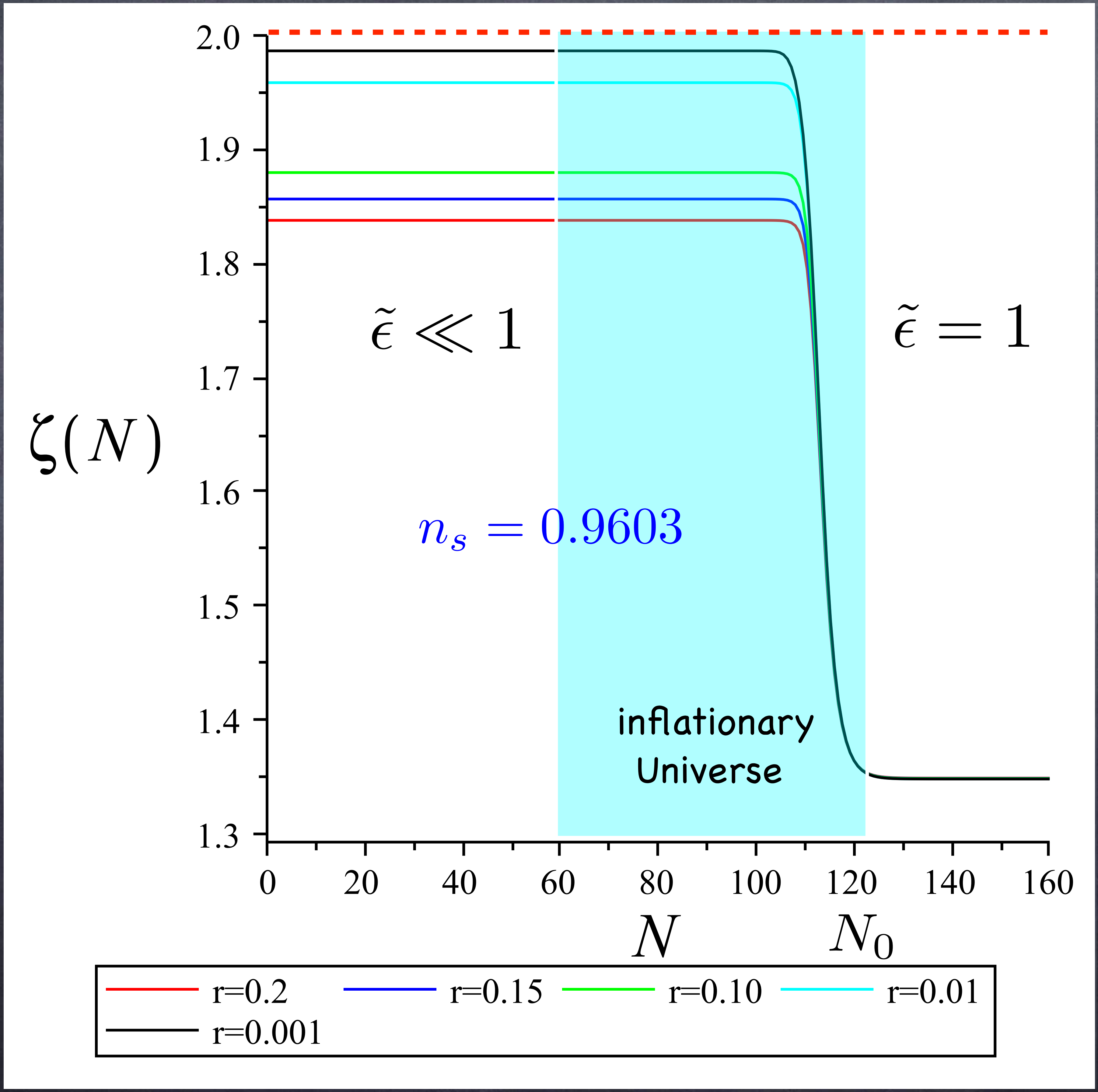
Slow-roll parameters in terms of the function

$$n_S - 1 = \frac{16C(3\alpha+2)\alpha'^2}{\alpha^2(\alpha+2)^4} - \frac{16(2+2\alpha+(3C+5)\alpha^2)\alpha'}{\alpha(\alpha+2)^4} - \frac{16C\alpha''}{\alpha(\alpha+2)^3} - \frac{24\alpha^2(\alpha^2+\alpha+1)}{(\alpha+2)^4}$$

$$r = \frac{48\alpha^2}{(\alpha+2)^2} - \frac{32\left(10\,\pi^2 - 24\,C^2 + 36\,C^2\alpha - 96 - 3\,\pi^2\alpha\right)\alpha'^2}{(\alpha+2)^6} +$$
$$+ \frac{192\alpha\left(8\,C + 8\,C\alpha - 3\,\pi^2\alpha^2 + 8\,C\alpha^2 + 30\,\alpha^2\right)\alpha'}{(\alpha+2)^6} + \frac{32\alpha\left(12\,C^2 - \pi^2\right)\alpha''}{(\alpha+2)^5}$$

Fix $n_s = 0.9603$, use r as parameter and solve for $\alpha(N)$

$$f(R) \sim R^{\zeta}$$



$\zeta(N)$
2.0
1.9
1.8
1.7
1.6
1.5
1.4
1.3
0
20
40
60
80
100
120
140
160
N
N_0
$\tilde{\epsilon} \ll 1$
$\tilde{\epsilon} = 1$
$n_s = 0.9603$
inflationary Universe
r=0.2
r=0.15
r=0.10
r=0.01
r=0.001

It has been proven that for the Euclidean action

$$I_E[g] = -\int d^4x\,\sqrt{\det(g)}\,f(R) = -\frac{b}{2}\int d^4x\,\sqrt{\det(g)}\,R^2$$

The 1-loop quantum corrections around de Sitter space amounts to [G.Cognola et al. JCAP02(2005)010]

$$\boxed{L = -\frac{b_0}{2}\,R^2\left(1+\frac{\gamma}{b_0}\ln\frac{R^2}{\mu_0^2}\right)}, \qquad b(\mu) = \gamma\,\ln\left(\frac{\mu^2}{\mu_0^2}\right) + b_0\,, \quad \gamma = -\frac{25}{1152\pi^2}$$

For $\gamma \ll b_0$:

$$L \sim -\frac{b_0}{2}\,R^2\left(\frac{R^2}{\mu_0^2}\right)^{\frac{\gamma}{b_0}} \equiv f_0 R^{\zeta}$$

However one can show that for this form

$$\boxed{n_s \simeq 1 + r/8}$$

Should we throw away log corrections? Maybe not if sum!

[MR et al PRD 91, 123527, 2015]

$$\boxed{f(R) = \frac{R^2}{1+\gamma \ln\left(R^2/\mu^2\right)}}$$

We cannot prove this formula but we can motivate it:

$$L = R^2 + \xi R\phi^2 - \frac{1}{2}(\partial\phi)^2 - \frac{1}{4}\lambda\phi^4(1+\delta\ln(\phi^2/\mu))$$

Non-minimally coupled Higgs with Coleman Weinberg terms.

<u>During slow-roll</u> we ignore kinetic terms so e.o.m. gives

$$2\xi R\phi - \frac{1}{2}\lambda\phi^3(2+\delta+2\delta\ln(\phi^2/\mu)) \simeq 0$$

Solve for the scalar field (Lambert function), expand and find

$$\phi^2 \sim \frac{R}{(1+\delta\ln(R^2/\mu^2)+\ln(\ln(R^2/\mu^2))+\dots)}$$

Slow-roll parameters for f(R) theories:

$$\epsilon = \frac{(FR-2f)^2}{3(FR-f)^2} \qquad \xi^2 = \frac{4(XR-2f)(X^3X'' + X'^3XR - 8X'^3 f + 3X^2X'^2)}{9X'^3(XR-f)^2}$$

$$\eta = \frac{2(FR-4f)}{3(FR-f)} + \frac{2F^2}{3FR - fX'} \qquad n_s \simeq 1 - 6\epsilon + 2\eta, \quad r = 16\epsilon$$

When applied to our model

$$f(R) = \frac{R^2}{1+\gamma \ln\left(R^2/\mu^2\right)}$$

we find

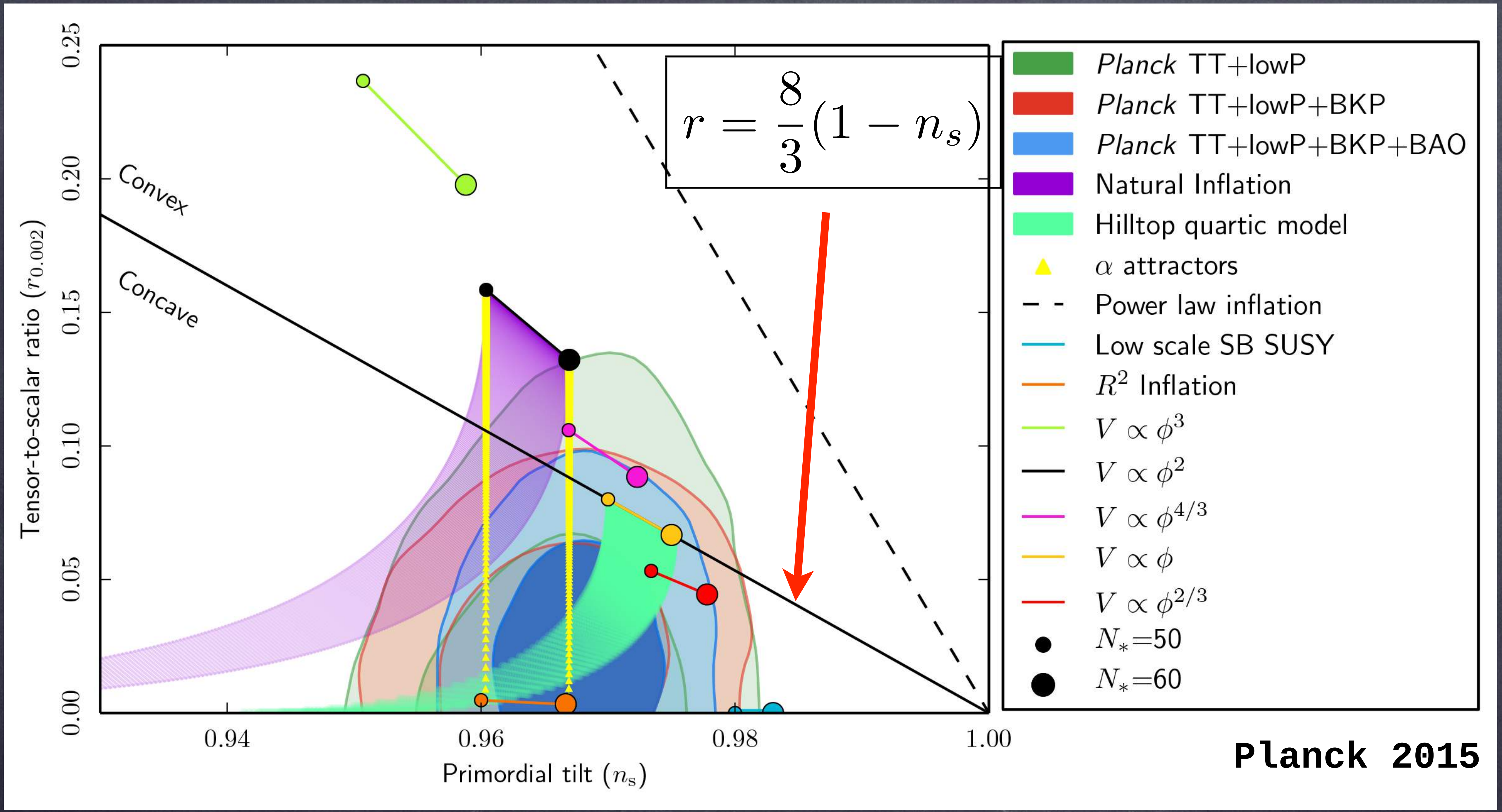
$$\boxed{r = \frac{8(1-n_s)}{3}}$$

<u>independent of all parameters</u>! The amplitude instead:

$$A_s = \frac{\mu(1+\gamma z)^2(1+\gamma z - 2\gamma)^3}{512\, M^2 \pi^2 \gamma^2 (1+\gamma z - \gamma)^2} \qquad z \equiv \ln\left(\frac{R^2}{\mu^2}\right)$$

evaluated at the horizon exit. Comparing with data we find

$$N = 45\,, \quad \gamma = 0.087 \qquad \sqrt{2\mu} \simeq 5 \times 10^{-6} M_p$$



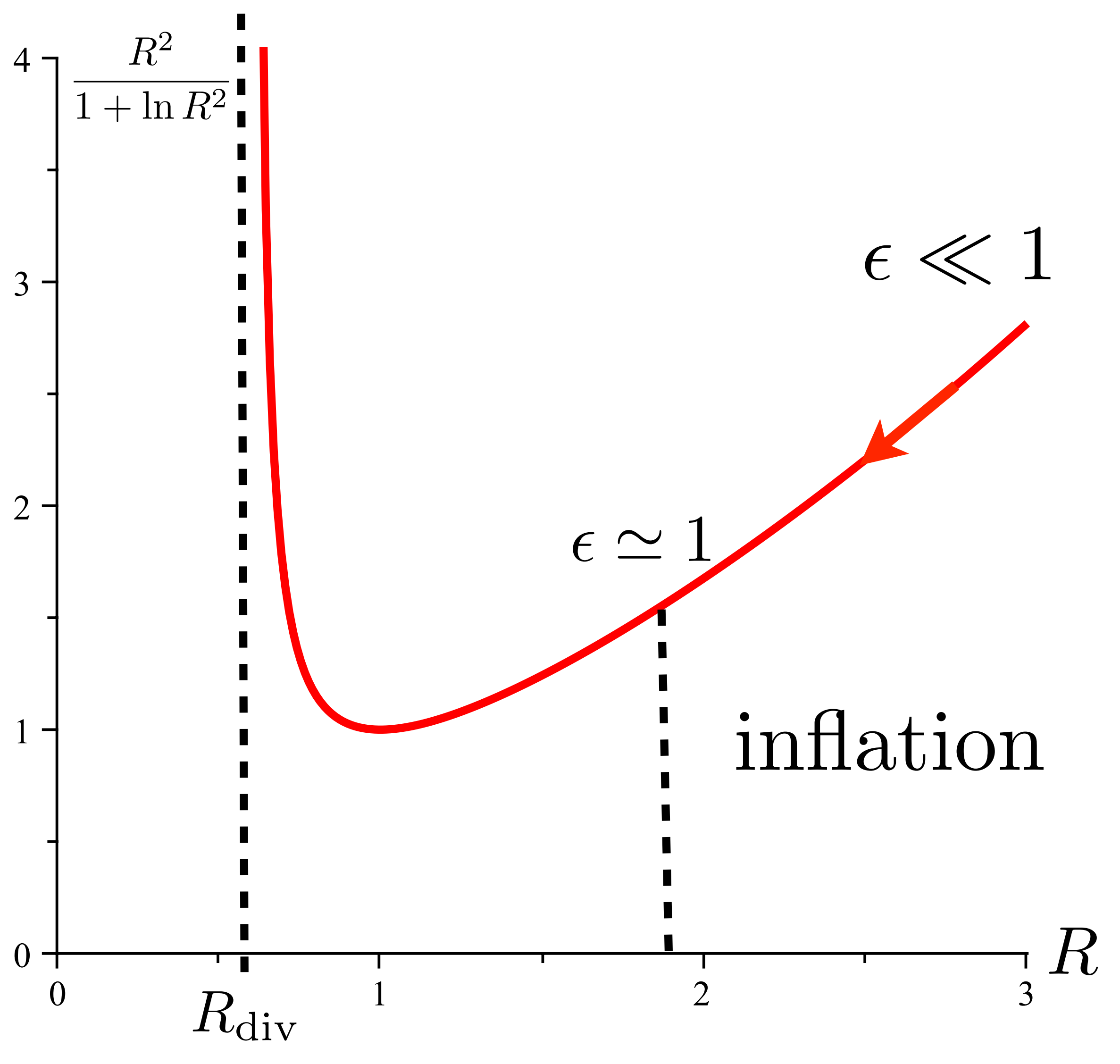
$$N = 40 \Rightarrow n_s = 0.966,\ \ r = 0.084,\ \ dn_s/d\ln k = -0.0008$$

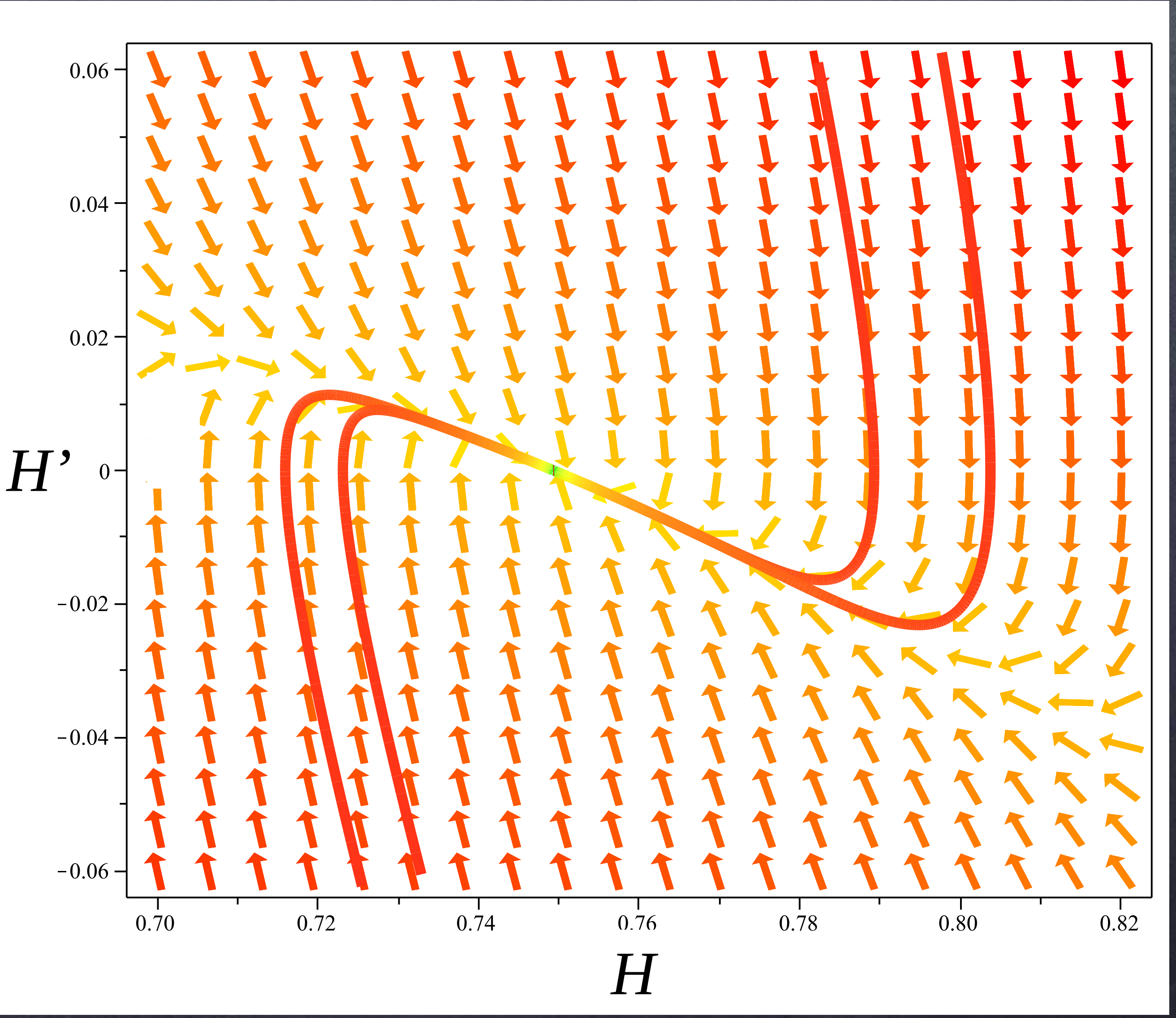
$$N = 50 \Rightarrow n_s = 0.973,\ \ r = 0.068,\ \ dn_s/d\ln k = -0.0005$$

Dynamical system evolution for the model

$$\boxed{f(R) = \frac{R^2}{1 + \gamma \ln\left(R^2/\mu^2\right)}}$$

There is one stable attractor corresponding to the pole





Spontaneous and classical breaking of scale-invariance

[MR & L. Vanzo, PRD 94, 024009, 2016]
[G. Tambalo & MR, 1610.06478]

Scale-invariant scalar tensor theory

$$\mathcal{L}_{\rm inv} = \sqrt{|\det g|}\left[\frac{\alpha}{36}R^2 + \frac{\xi}{6}\phi^2 R - \frac{1}{2}(\partial\phi)^2 - \frac{\lambda}{4}\phi^4\right]$$

effective potential for fixed R:

$$V_{\rm eff} = -\frac{\xi}{6}\phi^2 R + \frac{\lambda}{4}\phi^4$$

Max and min corresponds to fixed points:

$$\phi = 0\,, \qquad \phi_0^2 = \frac{\xi R}{3\lambda}$$

EOM in terms of efolding $N = \ln a$

$$H^2\phi'' + (HH' + 3H^2)\phi' - 2\xi\phi HH' - \phi(4\xi H^2 - \lambda\phi^2) = 0\,,$$

$$\alpha H^2\left(2HH'' + H'^2 + 6HH'\right) + 2\xi H^2\phi\phi' - \frac{1}{2}\phi'^2 H^2 + \frac{\phi^2}{4}(4\xi H^2 - \lambda\phi^2) = 0\,.$$

There are only two fixed points:

$(H, \phi) = (H, 0)$ H = arbitrary, saddle point

$(H, \phi) = (H, 2\sqrt{\frac{\xi}{\lambda}} H)$ H = arbitrary, stable point

We used the condition $\alpha = \xi^2/\lambda$

Which implies that, at the stable fixed point:

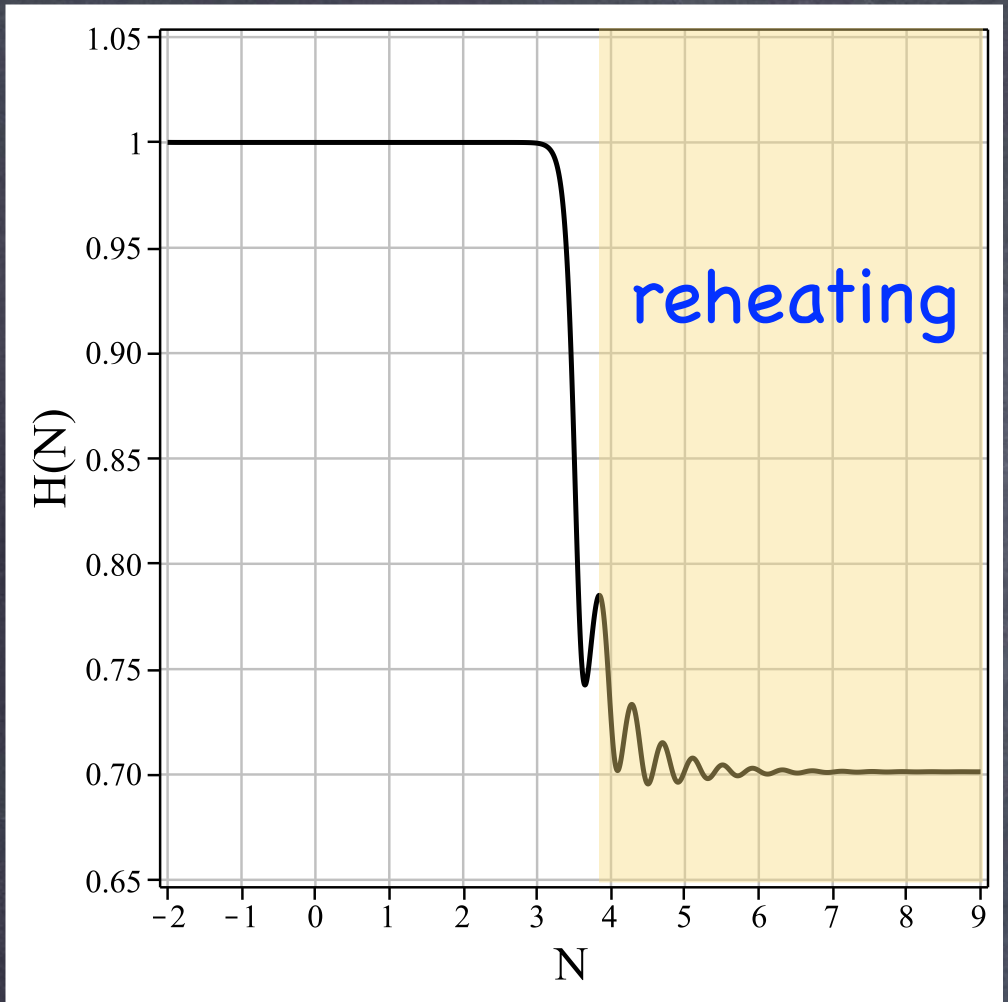
$$\Lambda_{\text{eff}} \equiv \frac{\alpha R^2}{36} - \frac{\lambda\phi^4}{4} \to 0$$

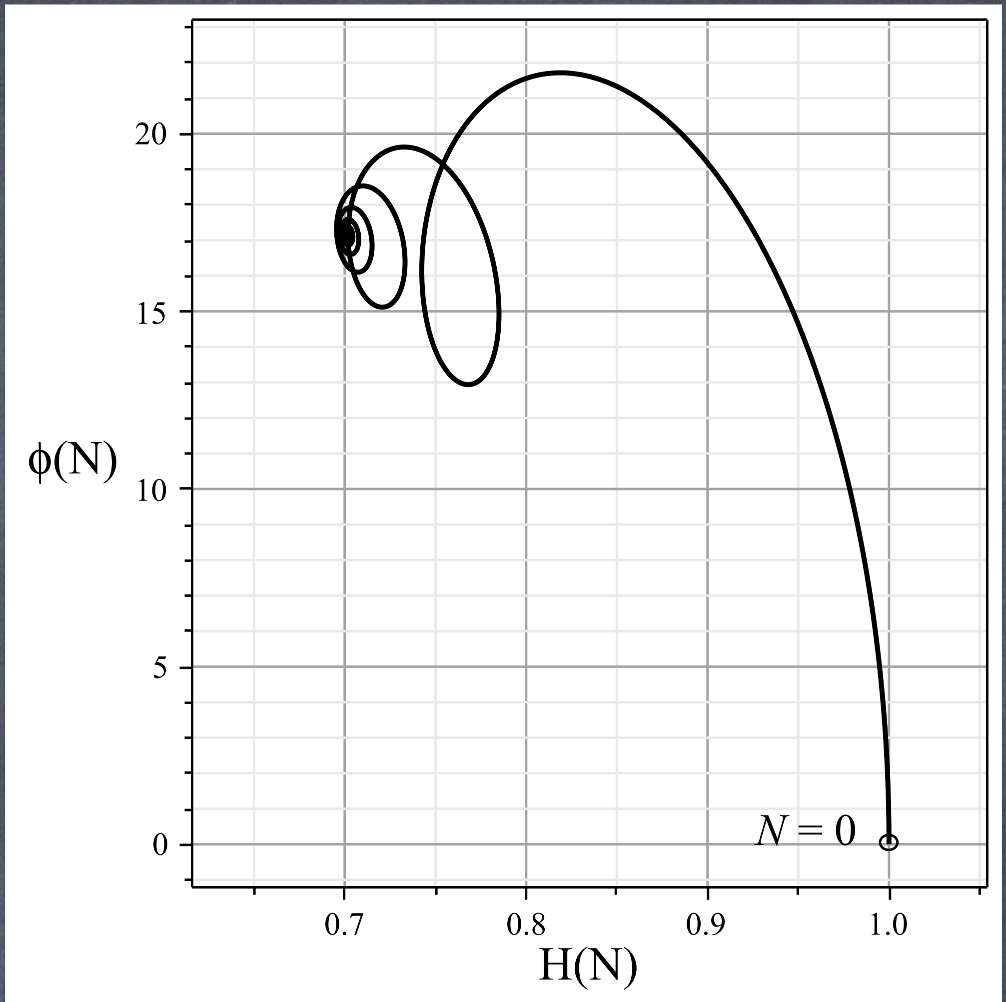
Recall:

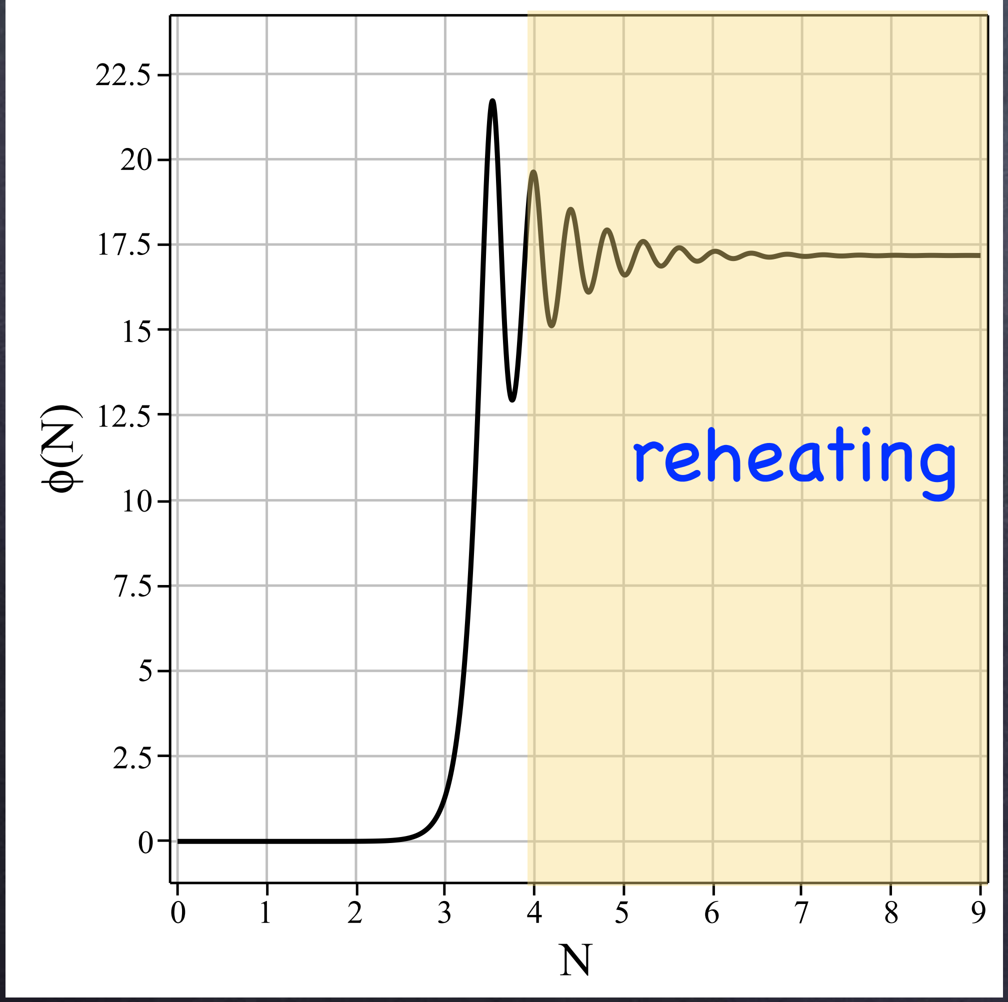
$$\mathcal{L}_{\text{inv}} = \sqrt{|\det g|}\left[\frac{\alpha}{36}R^2 + \frac{\xi}{6}\phi^2 R - \frac{1}{2}(\partial\phi)^2 - \frac{\lambda}{4}\phi^4\right]$$

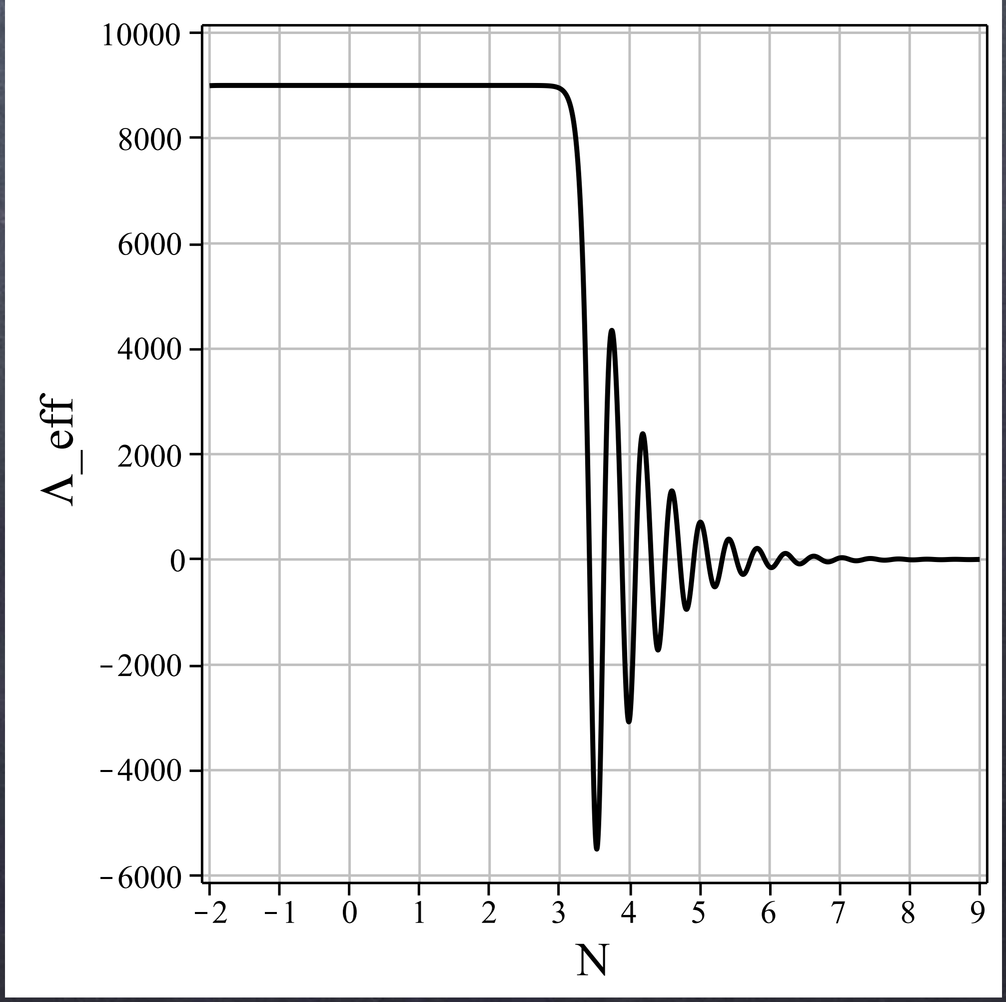
At the stable fixed point the scalar field is stabilised and the mass scale emerges. Symmetry is spontaneously broken

$$\boxed{M = \sqrt{\xi/3}\,\phi_0 \equiv m_p}$$


H(N)
N
reheating
1.05
1
0.95
0.90
0.85
0.80
0.75
0.70
0.65
−2 −1 0 1 2 3 4 5 6 7 8 9


ϕ(N)
H(N)
N = 0
20
15
10
5
0
0.7 0.8 0.9 1.0


ϕ(N)
N
reheating
22.5
20
17.5
15
12.5
10
7.5
5
2.5
0
0 1 2 3 4 5 6 7 8 9


Λ_eff
N
10000
8000
6000
4000
2000
0
−2000
−4000
−6000
−2 −1 0 1 2 3 4 5 6 7 8 9

The Hubble parameter stabilises at $H_\star^2 = \frac{\Lambda}{3}$

It cannot be the present cosmological constant. It would require a huge fine-tuning for the other parameters since:

$$\Lambda = \frac{\lambda \phi_0^4}{4M_p^2}$$

Rather, $H_\star$ can be seen as the initial condition for the subsequent reheating phase. We can write

$$H_\star = m_p \sqrt{3\lambda}/(2\xi)$$

and for $H_\star \sim 10^{14}$ GeV and $\xi \sim 1$ we find $\lambda \sim 10^{-8}$

Also, the e-folding number is fixed by the initial condition

$$H_{in}/\phi_{in} \sim \exp(\Delta N - 9)$$

For 50/60 efolds inflation needs to begin very close to the unstable fixed point

Reheating and/or Preheating scenarios

Reheating via scalar field decay $\phi \to \chi + \chi$. Assume that

$$\boxed{\mathcal{L}_{\rm tot} = \mathcal{L}_{\rm inv} - g^2\phi^2\chi^2 - \frac{1}{2}(\partial\chi)^2 - \frac{1}{2}m_\chi^2\chi^2}$$

Around the fixed point with we replace $\phi \to \phi - \phi_0$

$$\mathcal{L}_{\rm reh} \simeq -\frac{1}{2}m_\phi^2\phi^2 + 2g\phi_0\phi\chi^2 + \dots \qquad m_\phi^2 = \frac{3\lambda\phi_0^2}{2} - \frac{\xi\langle R\rangle}{3} \simeq \frac{\lambda\phi_0^2}{2}$$

the decay rate can be estimated as

$$\Gamma = \frac{g^2\phi_0^2}{8\pi m_\phi} = \sqrt{\frac{2}{\lambda}}\frac{g^2\phi_0}{8\pi}$$

The process stops at $H \sim \Gamma$ at reheating temperature of

$$\boxed{T_{\rm reh} \simeq \sqrt{\Gamma M_p} \simeq 0.3 \times g M_p(\lambda\xi)^{-1/4}}$$

There are <u>two pre-heating channels</u> via parametric resonance. Take a massless scalar field minimally coupled:

$$\ddot{\chi}_k + 3H\dot{\chi}_k + \left(\frac{k^2}{a^2} + g^2\phi^2\right)\chi_k = 0$$

usual oscillator with time-dependent frequency

$$\omega_k = \left(\frac{k^2}{a^2} + g^2\phi^2\right)^{\frac{1}{2}}$$

Adiabaticity parameter: $\mathcal{A} = \left|\dot{\omega}/\omega^2\right|$ when small no particle production. We have:

$$\mathcal{A} \simeq \left|\frac{\phi' H}{g\phi^2}\right|$$

since the scalar field oscillates we expect large burst of particles when the field becomes small. This is the typical case of parametric resonance.

But there is another channel!

Introduce the variable $X_k = a^{3/2}\chi_k$

The occupation number is $n_k = \frac{\omega_k}{2}\left(\frac{|\dot{X}_k|^2}{\omega_k^2} + X_k^2\right) - \frac{1}{2}$

The Klein-Gordon equation becomes

$$X_k'' + \frac{H' X_k'}{H} + \left(\frac{k^2}{a^2 H^2} + \frac{g^2\phi^2}{H^2}\right) X_k = 0$$

We can solve the eom near the stable fixed point

$$H(N) = H_0 + T(N)\, e^{-\frac{3}{2}N}$$

$$\phi(N) = \phi_0 - \gamma\, T'(N) e^{-\frac{3}{2}N}$$

$$T(N) = c_3 \sin(KN) + c_4 \cos(KN)\,, \quad \gamma = \xi(1+2\xi)^{-1}\sqrt{\xi/\lambda}$$

Then:

$$X_k'' + \left(\frac{T'}{T} - \frac{3}{2}\right) X_k' + \left(P^2 + g^2\gamma^2 \frac{T'^2}{T^2}\right) X_k = 0$$

$P = k/(aH) \ll 1$

We have two interesting regimes for the solutions of

$$X_k'' + \left(\frac{T'}{T} - \frac{3}{2}\right) X_k' + \left(P^2 + g^2\gamma^2 \frac{T'^2}{T^2}\right) X_k = 0$$

ϕ - Amplification: when $|T'/T| \ll 1$ then

$$X_k \sim \exp\left(\frac{3}{4} \pm \frac{1}{4}\sqrt{9 - 16P^2}\right) \qquad n_k \sim \frac{T(N)}{P} \exp\left(\frac{3}{2}N\right)$$

this is the usual case when $\mathcal{A} = \left|\dot{\omega}/\omega^2\right|$ becomes large.

H - Amplification: if/when H vanishes then (N_0 arbitrary)

$$X_k'' - \frac{1}{N - N_0} X_k' + \left(P^2 + \frac{g^2\gamma^2}{(N - N_0)^2}\right) X_k \simeq 0$$

Can be solved with Bessel's functions and:

$$X_k(N - N_0) \sim (N - N_0)^{1-\nu} \qquad \nu = \sqrt{1 - g^2\gamma^2}$$

The analysis can be repeated in the Einstein frame:

$$\mathcal{L}_E = \sqrt{\tilde{g}}\left[\frac{M^2}{2}\tilde{R} - \frac{1}{2}\tilde{g}^{\mu\nu}\partial_\mu\psi\partial_\nu\psi - \frac{1}{2}\exp\left(-\frac{\sqrt{2}\psi}{\sqrt{3}M}\right)\tilde{g}^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi,\psi) - \frac{9\lambda M^4}{4\xi^2}\right]$$

$$V(\phi,\psi) = \frac{\lambda\phi^4}{2}\exp\left(-\frac{2\sqrt{2}\psi}{\sqrt{3}M}\right) - \frac{3\lambda M^2\phi^2}{2\xi}\exp\left(-\frac{\sqrt{2}\psi}{\sqrt{3}M}\right)$$

The "scalaron":

$$\psi = \frac{1}{2}\sqrt{6}M\ln\left(\frac{\alpha R}{9M^2} + \frac{\xi\phi^2}{3M^2}\right)$$

Unstable fixed point at

$$H_{\rm unst} = \frac{\sqrt{3\lambda}M}{2\xi}\,, \quad \phi_{\rm unst} = 0\,, \quad \psi_{\rm unst} = \text{arbitrary}$$

Stable fixed point at

$$H_{\rm stab} = \frac{\sqrt{3\lambda}M}{2\sqrt{2}\xi}\,, \quad \phi_{\rm stab} = \phi_0 = \text{arbitrary}\,, \quad \psi_{\rm stab} \simeq 0.8489\,M$$

Note:

$$H_{\rm unst}/H_{\rm stab} = \sqrt{2}$$

Road Ahead

- Perturbations (CMG, grav. waves, non-gaussianities)
- Add other scale invariant fields (e.g. radiation)
- Preheating and/or reheating
- Understand quantum effects
- Is spatial curvature admissible?
- Bouncing solutions?

Conclusions

- Scale-invariance can be a fundamental symmetry of Nature.
- The symmetry breaking can occur via quantum effects or spontaneous (classical) symmetry breaking.
- Inflationary Universe is an ideal laboratory to test fundamental scale-invariance and its breaking.
- Work has been done in black hole thermodynamics too (MR et al, Entropy 17 (2015) 5145, PRD91 (2015) 104004)
- Many interesting questions arise in scale-invariant theories.

KIITOS!

Extra material

Quasi-scale invariant attractors

(MR et al, PRD 93, 024040 (2016))

We consider the non-canonical Linde-like Lagrangian *(p > 0)*

$$L = \sqrt{-g}\left(\frac{M^2}{2}R - \frac{A_p}{2\phi^p}(\partial\phi)^2 - V(\phi)\right)$$

The the first Hubble flow parameters are

$$\epsilon_1 \simeq \frac{M^2}{2}\frac{\phi^p}{A_p}\left(\frac{V_\phi}{V}\right)^2,$$

$$\epsilon_2 \simeq 4\epsilon_1 - \frac{M^2\phi^{p-1}}{A_p}\left(p\frac{V_\phi}{V} + 2\phi\frac{V_{\phi\phi}}{V}\right)$$

Then we have:

$$n_s \simeq 1 - 2\epsilon_1 - \epsilon_2,$$

$$r \simeq 16\epsilon_1\,.$$

Quasi-scale invariant attractors

The alpha-attractors required an analytic potential at the pole.
Here we consider non-analytic potentials:

$$V(\phi) = V_0 \ln\left(\frac{\phi}{\phi_0}\right) , \quad p = 2$$

$$V(\phi) = V_0 \left(\frac{\phi}{\phi_0}\right)^{(2-p)/2} , \quad p > 2$$

At leading order, these yield

$$\epsilon_2 \simeq 4\epsilon_1 \quad \Rightarrow \quad \boxed{r = \frac{8}{3}(1 - n_s)}$$

That is the same as for $\quad f_{\rm eff}(R) = \dfrac{R^2}{\left[1 + \gamma \ln\left(\frac{R^2}{\mu^2}\right)\right]}$

Is this just a coincidence?

Transform to Jordan frame: $L = \sqrt{-g}\left(\frac{M^2}{2}R - \frac{A_p}{2\phi^p}(\partial\phi)^2 - V(\phi)\right)$

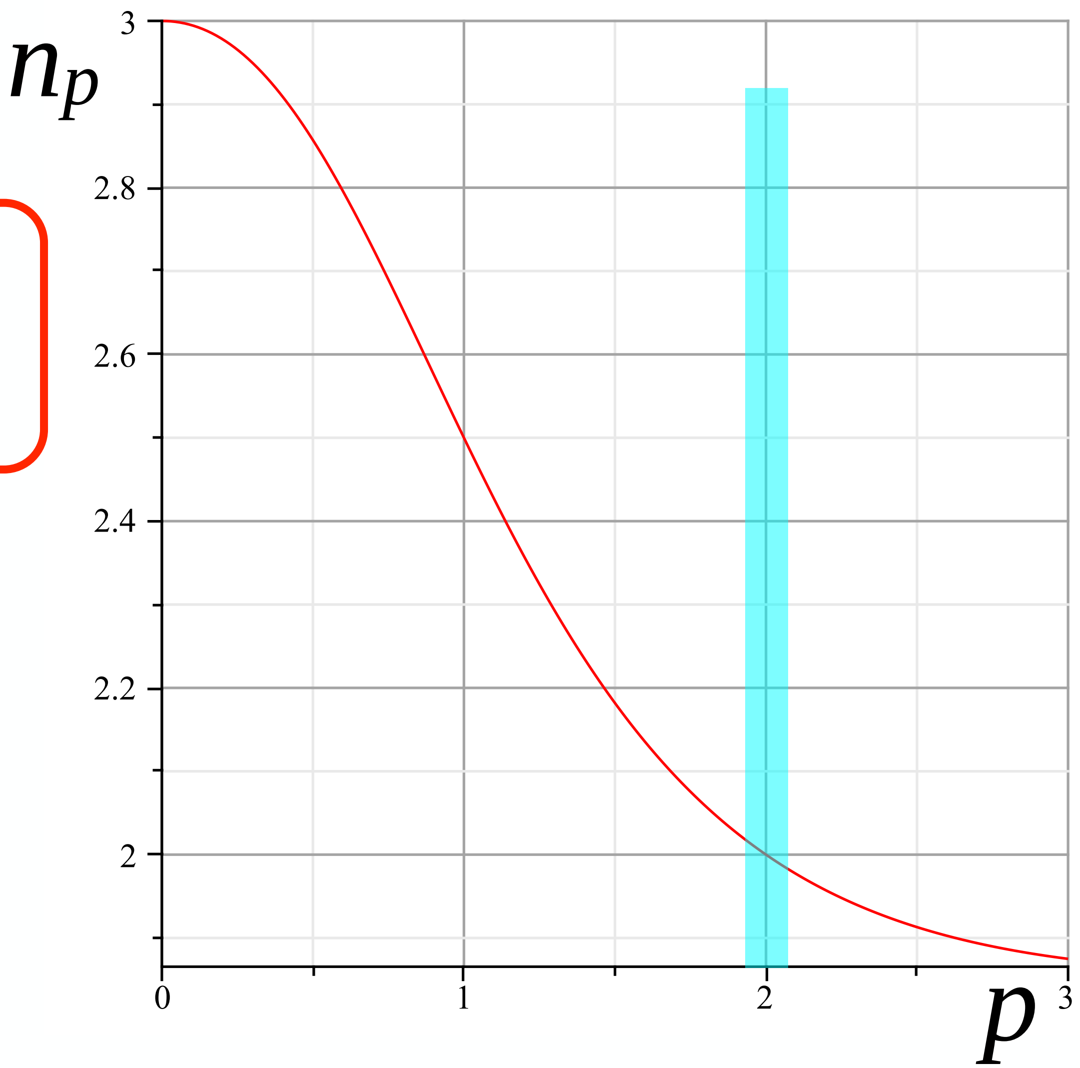
$g_{\mu\nu} \to A_p^{1/2}\phi^{p/2}g_{\mu\nu}$

$$\boxed{\begin{aligned} &L_J = \sqrt{-g}\left[\xi\phi^2 R - \frac{1}{2}(\partial\phi)^2 - \lambda\phi^4\left(1-\gamma\ln\frac{\phi^2}{\mu^2}\right)\right] && p=2 \\ &L_J = \sqrt{-g}\left[\xi_p\phi^p R - \frac{1}{2}(\partial\phi)^2 - \lambda_p\phi^{2p+\frac{2}{p}-1}\right] && p\neq 2 \end{aligned}}$$

Go on-shell in slowroll approx:

$$\boxed{f(R) = \sqrt{-g}\,\alpha R^2\left(1+\gamma\ln\frac{R}{\mu^2}\right)^{-1} \qquad p=2}$$

$$\boxed{\begin{aligned} &L \sim \sqrt{g}R^{n_p} \\ &n_p = 2 + \frac{2-p}{p^2-p+2} && p\neq 2 \end{aligned}}$$



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Non Canonical
E-frame

$$\boxed{\frac{1}{\phi^p}(\partial\phi)^2 + V}$$

analytic V

α-attractors

non-analytic V

Canonical
E-frame

$$\boxed{(\partial\psi)^2 + k\psi}$$

sharp prediction

$$r = \frac{8}{3}(1 - n_s)$$

J-frame
Induced gravity
Coleman-Weimberg
corrections

on-shell

slowroll

quasi-scale invariant

$$\boxed{R^2 + \text{corrections}}$$

Generalizations

Combination of power-law and logarithmic poles:

$$L = \frac{A_p}{\phi^p}(\partial\phi)^2 + \phi^{(2-p)/p}\left[1 + a\ln\left(\frac{\phi}{m}\right)\right] \qquad p \neq 2, \quad |a| \ll 1$$

We find: $$\boxed{r \simeq \frac{8}{3}(1-n_s) - \frac{32(1-n_s)a}{9(p-2)} + \mathcal{O}(a^2)}$$