Spectral dimension and other ways to recover geometric information from spectral triples

Lisa Glaser

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universität wien

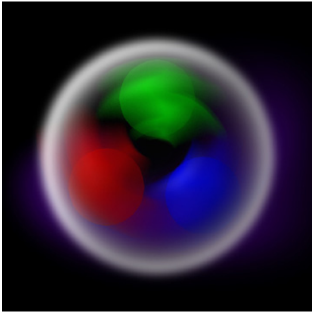
Plan for this talk:

- Spectral triples
- Fuzzy spaces
- Spectral observables
- Visualizing Truncated triples

Quantum Gravity on the computer

Computer simulations can:

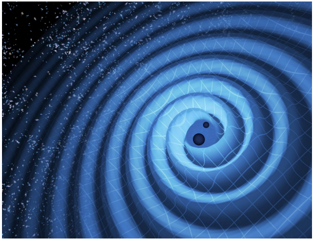
- Calculate the mass of the Proton (lattice QCD)



Quantum Gravity on the computer

Computer simulations can:

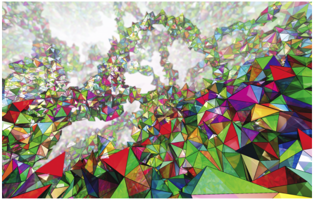
- Calculate the mass of the Proton (lattice QCD)
- Predict gravitational wave signatures of colliding neutron stars and black holes (numerical relativity)



Quantum Gravity on the computer

Computer simulations can:

- Calculate the mass of the Proton (lattice QCD)
- Predict gravitational wave signatures of colliding neutron stars and black holes (numerical relativity)
- Maybe they can also help us understand Quantum Gravity?



The path integral of Quantum Gravity

$$\langle f \rangle = \frac{\int f(g)\, e^{i\mathcal{S}(g)}\, \mathcal{D}[g]}{\int e^{i\mathcal{S}(g)}\, \mathcal{D}[g]}$$

Ingredients:

- Geometry g and measure $\mathcal{D}[g]$
- Functions of geometry f
- Action $\mathcal{S}$

The path integral of Quantum Gravity

$$\langle f \rangle = \frac{\int f(g)\, e^{i\mathcal{S}(g)}\, \mathcal{D}[g]}{\int e^{i\mathcal{S}(g)}\, \mathcal{D}[g]}$$

Ingredients:

- Geometry g and measure $\mathcal{D}[g]$
  - Use spectral triples to define g and $\mathcal{D}[g]$
- Functions of geometry f
- Action $\mathcal{S}$

The plan

- Spectral triples
- Fuzzy spaces
- Spectral observables
- Visualizing Truncated triples

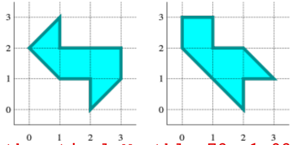
Motivation: Can we hear the shape of a drum?

Eigenvalue problem:
For a membrane Ω held fixed along bdry Γ the eigenvalue problem can be stated as:

$$\frac{1}{2}\nabla^2\psi_n(x) + \lambda_n\psi_n(x) = 0$$

$$\psi_n(x) = 0 \text{ on } \Gamma$$

If two membranes Ω_1, Ω_2 (boundaries Γ_1, Γ_2) lead to the same spectrum λ_n, are they the same (up to symmetry transformations)?



(M. Kac, The American Mathematical Monthly 73, 1-23 (1966))

Geometry as a spectral triple

$$(\mathcal{A}, \mathcal{H}, D)$$

- an Algebra $\mathcal{A}$ with action on $\mathcal{H}$
- a Hilbert space $\mathcal{H}$
- a Dirac operator D acting on $\mathcal{H}$

The 'sound' is not enough

- $\{\lambda_n\} = spec(D)$ is not enough, we need to know $\mathcal{A}$, algebra of functions on the space (drum)
- For manifolds $\mathcal{A}$ is commutative, can generalize to non-commutative

(A. Connes, Int.J.Geom.Meth.Mod.Phys. 5, 1215-1242 (2008))

(more detail e.g. A. Connes, Commun.Math.Phys. 182, 155-176 (1996))

Geometry as a spectral triple

$(\mathcal{A}, \mathcal{H}, D)$

- an Algebra $\mathcal{A}$ with action on $\mathcal{H}$
- a Hilbert space $\mathcal{H}$
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Axioms of non-commutative geometry [a]

- $\exists$ a faithfull action $\mathcal{A}$ in $\mathcal{H}$
- $\mathcal{H}$ is a bimodule over $\mathcal{A}$ (there is a left and a right action of $\mathcal{A}$ in $\mathcal{H}$)
- First order condition $[[D, a\triangleright], \triangleleft b] = 0$ for $a, b \in \mathcal{A}$

[a] Abridged version

(A. Connes, Int.J.Geom.Meth.Mod.Phys. 5, 1215-1242 (2008))

(more detail e.g. A. Connes, Commun.Math.Phys. 182, 155-176 (1996))

A simple geometry as a spectral triple

The circle as an algebra with a unitary operator U acting on $\mathcal{H} = L^2(\mathbb{S}^1)$

$$
\begin{array}{rcl}
UU^* = 1 & & D = D^* \\
U^*[D, U] = 1 & \leftrightarrow & U^* D U = D + 1 \\
De_n = \lambda_n e_n & \leftrightarrow & DUe_n = (\lambda_n + 1) U e_n
\end{array}
$$

U generates the algebra

$$a = \sum_{\mathbb{Z}} a_n U^n$$

$$a_n \in \mathbb{C}$$

for any algebra element a

Spectral triple

$$(C^\infty(\mathbb{S}^1), L^2(\mathbb{S}^1), -i\partial_\phi)$$

A simple geometry as a spectral triple

For a commutative torus take two S^1 generators U, V

$$U^* U = V^* V = 1$$

We can make the torus non-commutative by introducing

$$UV = \vartheta VU \qquad \vartheta = e^{2\pi i\theta}$$

U, V generate the algebra

$$a = \sum_{\mathbb{Z}^2} a_{n,m} U^n V^m$$

for any algebra element a

Spectral triple

$$(C^\infty(\mathbb{T}^2), L^2(\mathbb{T}^2), -i\sigma^j\partial_j)$$

with σ_j the two off diagonal pauli matrices

Spectral triples as quantum geometry?

How about using spectral triples as quantum geometry?

I prefer my space-time discrete/ finite, so there are two options:

Fuzzy spaces:

- Works best for very symmetric spaces
- Uses finite $\mathcal{A}$
- finds D to respect first order condition

Truncations of spectral triples:

- Any spectral triple (they are mostly still very symmetric)
- truncates $\mathcal{A}$ and $\mathcal{H}$
- break first order condition

Fuzzy space (p, q)

$(s, \mathcal{H}, \mathcal{A}, \Gamma, J, \mathcal{D})$

- The algebra are matrices:
  $\mathcal{A}$ is a $*-$ algebra $M(n, \mathbb{C})$
- Acting on a Hilbert space:
  $\mathcal{H} = V \otimes M(n, \mathbb{C})$
  where V is a (p, q)-Clifford module

Extra ingredients to make it a real spectral triple

- KO-dimension;
  $s = (q - p) \bmod 8$
- Chirality;
  $\Gamma(v \otimes m) = \gamma v \otimes m$ with γ the chirality operator on V
- Real structure;
  $J(v \otimes m) = Cv \otimes m^*$ where C is charge conjugation on V
  $J : \mathcal{H} \to \mathcal{H}$ with $\langle Ju, Jv \rangle = \langle u, v \rangle$

(as stated in J.W. Barrett J.Math.Phys. 56, 082301 (2015))

Fuzzy sphere

The continuum sphere

$$\left(\mathcal{A} = SU(2), \mathcal{H} = L^2(S^2, S), D = \sigma^\mu(\partial_\mu + \omega_\mu)\right)$$

with σ^μ the Pauli matrices and ω_μ a spin connection.

The fuzzy sphere is a finite spectral triple that approximates this.

- $\mathcal{A} = M(n, \mathbb{C})$
  with the irred. reps. of $SU(2)$ up to spin j, with $j = \frac{1}{2}(n-1)$ a natural basis,
- $\mathcal{H} = \mathbb{C}^4 \otimes M(n, \mathbb{C})$
- $\mathcal{D} = 1 + \sum_{j<k}^{3} \sigma^j \sigma^k \otimes [L_{jk}, \cdot]$
  with L_{jk} the lie algebra generators of $so(3)$ and σ^j the Pauli matrices.

Dirac operator : Form

Conditions on $\mathcal{D}$ for a real spetral triple

$$\mathcal{D} = \mathcal{D}^\dagger \qquad \mathcal{D}\Gamma = \pm\Gamma\mathcal{D}$$

$$\mathcal{D}J = \pm J\mathcal{D} \qquad [[\mathcal{D}, \rho(a)\triangleright], \triangleleft\rho(b)] = 0$$

Can be translated for a fuzzy space to:

$$\mathcal{D}(v \otimes m) = \sum_i \omega^i v \otimes \Big(\overbrace{K_i m}^{\text{left action}} + \epsilon' \overbrace{m K_i^*}^{\text{right action}} \Big)$$

(J.W. Barrett, J.Math.Phys. 56, 082301 (2015).)

Explore path integral over fuzzy spaces

$$\langle f \rangle = \frac{\int f(D) e^{-\mathcal{S}(D)} \mathrm{d}D}{\int e^{-\mathcal{S}(D)} \mathrm{d}D}$$

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$$\langle f \rangle = \frac{\int f(D) e^{-\mathcal{S}(D)} \mathrm{d}D}{\int e^{-\mathcal{S}(D)} \mathrm{d}D}$$
$$= \frac{\int f(D(K_i)) e^{-\mathcal{S}(D(K_i))} \prod_i \mathrm{d}K_i}{\int e^{-\mathcal{S}(D(K_i))} \prod_i \mathrm{d}K_i}$$

The simplest action

$$\mathcal{S} = g_2 \mathrm{Tr}\left(\mathcal{D}^2\right) + \mathrm{Tr}\left(\mathcal{D}^4\right)$$

(J.W. Barrett, LG J.Phys. A49, 245001 (2016))

What do we want from an action?

- physical motivation
  $\Rightarrow$ lowest order when expanding a heat kernel
- bounded from below
  $\Rightarrow$ for some g_2
- rises fast to infinity
  $\Rightarrow$ to make simulations possible

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$\Rightarrow$ Matrix model in K_i

Look for phase transitions

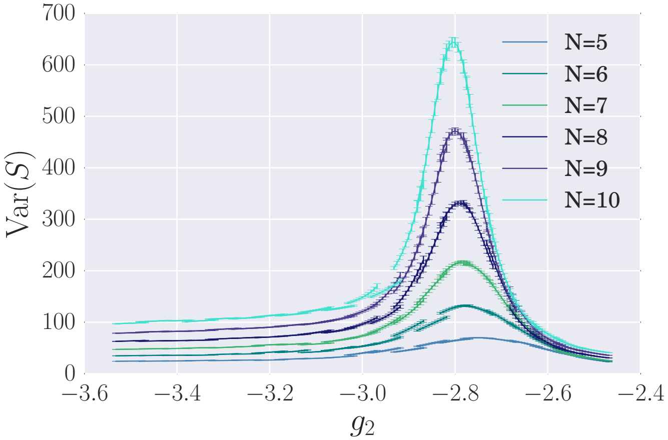
Phase transition

- qualitative change in behavior
- Phase transition marked by peak in Variance

$$Var(\mathcal{S}) = \langle \mathcal{S}^2 - \langle \mathcal{S} \rangle^2 \rangle$$

- Gets sharper in larger systems
- Higher order phase transitions show signs of correlation

Look for phase transitions



$$g_c^{(2,0)} = -2.781 \pm 0.289$$

(LG J.Phys.A50, 275201 (2017))

A (biased) overview:

Done:

- First tests of model

  (J.W. Barrett, LG, J.Phys. A49, 245001 (2016))
- Scaling behavior of model

  (LG, J.Phys.A50, 275201 (2017))
- Spectral dimension (next section!)

  (J.W. Barrett, P.J. Druce, LG, J.Phys. A52 275203 (2019))

Work pending:

- Larger matrix sizes

  (w.i.p. M. D'Arcangelo, J.W. Barrett)
- Recognize geometry
- What is matter / include matter
- Analytic results

The plan

- Spectral triples
- Fuzzy spaces
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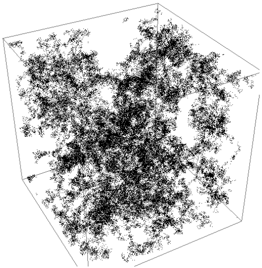
Spectral dimension



Return probability of random walk/ diffusion process aka. heat kernel

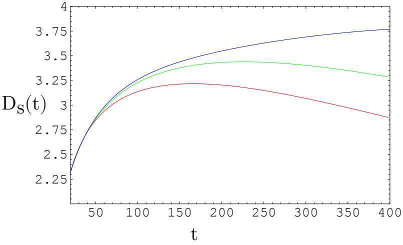
$$K(t) = \sum_i e^{-t\lambda_i^{(\Delta)}} \quad \lambda_i^{(\Delta)} \in Ev(\Delta)$$

$$\sim t^{-d/2}(a_0 + a_2 t + a_4 t^2 + \ldots)$$

Can determine dimension from the small t behaviour of K

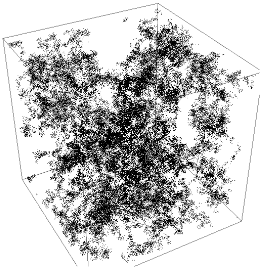
$$D_s(t) = \frac{\partial \log K(t)}{\partial \log(t)}$$

$$= t\, \frac{\sum_{\lambda^{(\Delta)}} \lambda^{(\Delta)} e^{-t\lambda^{(\Delta)}}}{\sum_{\lambda^{(\Delta)}} e^{-t\lambda^{(\Delta)}}}$$



$D_s(t)$ in CDT

Spectral dimension



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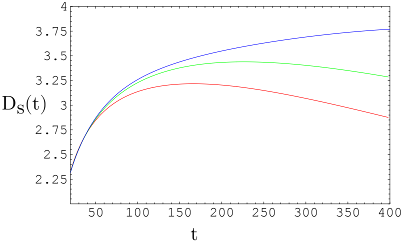
We have $\mathcal{D}$ not Δ!



$D_s(t)$ in CDT

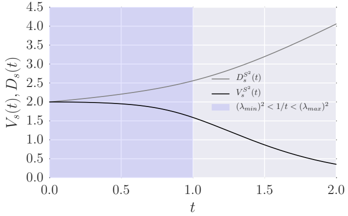
The spectral variance

Use $\mathcal{D}^2$ to get a Δ-type operator

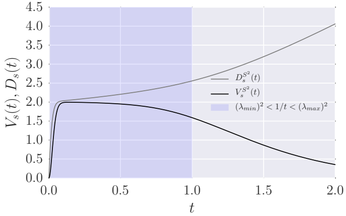
$$D_s(t) = 2t\,\frac{\sum_\lambda \lambda^2 e^{-t\lambda^2}}{\sum_\lambda e^{-t\lambda^2}}$$
$$\sim t\lambda_0^2 \quad \text{for large } t$$
$$\lambda \in Ev(\mathcal{D})$$

$\lambda_0^\Delta = 0 \Rightarrow$ no problem $\lambda_0 \neq 0 \Rightarrow$ linear mode

(J.W. Barrett, P.J. Druce, LG, J.Phys.A:Math.Theor.(2019))



exact S^2



fuzzy with $N = 10$

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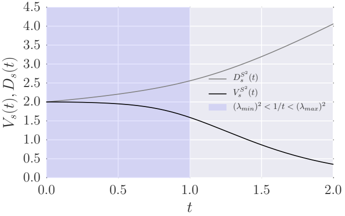
$$\sim t\lambda_0^2 \quad \text{for large } t$$

$$\lambda \in Ev(\mathcal{D})$$

$$V_s(t) = D_s(t) - t\frac{\partial D_s(t)}{\partial t}$$

(J.W. Barrett, P.J. Druce, LG, J.Phys.A:Math.Theor.(2019))



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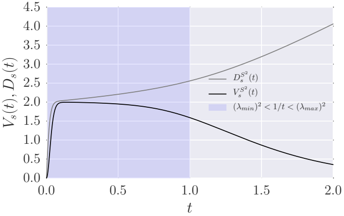
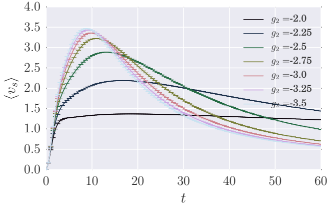
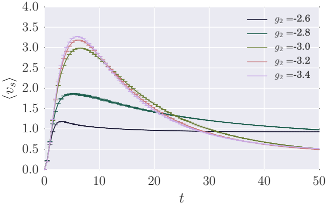


fuzzy with $N = 10$

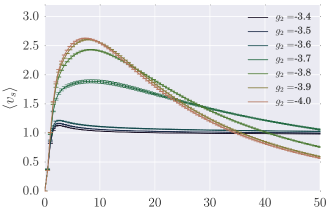
Spectral variance on random fuzzy spaces



$V_s(t) \quad (1,1)$

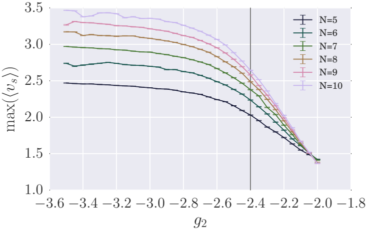


$V_s(t) \quad (2,0)$

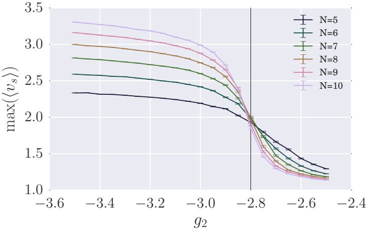


$V_s(t) \quad (1,3)$

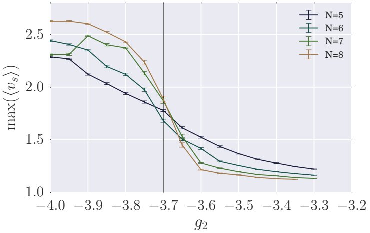
Maximum of the spectral variance



Type $(1,1)$ $N = 10$



Type $(2,0)$ $N = 10$



Type $(1,3)$ $N = 10$

Intersect at g_c !

Distance between geometries

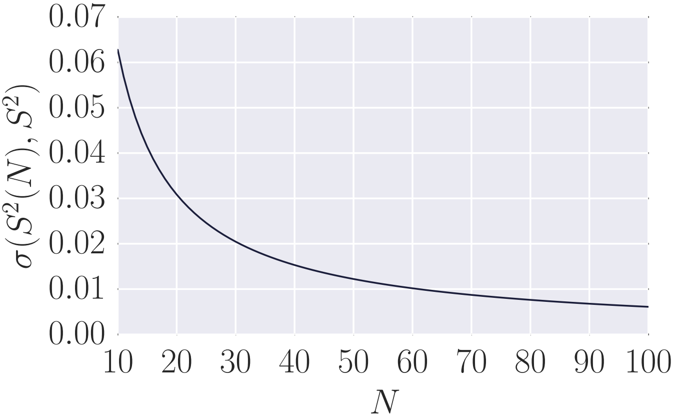
For two geometries X_1, X_2

$$d_{CK}(X_1, X_2) = \sup_{\gamma < s < \gamma+1} \left| \log \left| \frac{\zeta^{X_1}(s)}{\zeta^{X_2}(s)} \right| \right|$$

- $\gamma > d/2$ for X_1, X_2 in continuum, otherwise free

(Cornelissen, G. & Kontogeorgis, A. Lett Math Phys (2017) 107: 129)

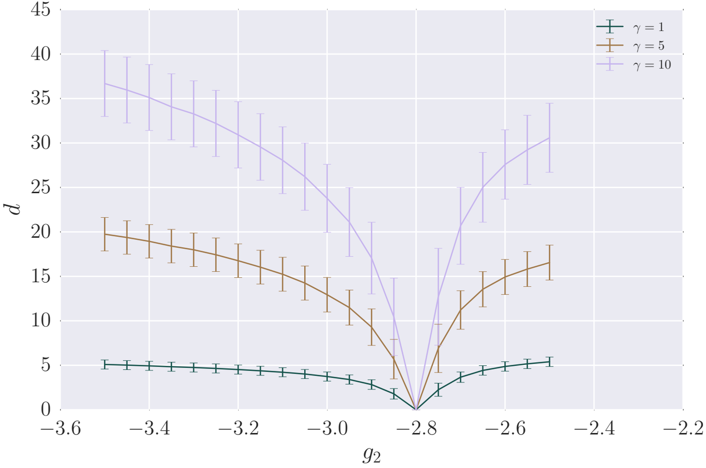
Testing the distance measure



Fuzzy S^2 to exact S^2

(J.W. Barrett, P.J. Druce, LG arXiv:1902.03590 accepted at J.Phys.A)

Testing the distance measure



Type $(2,0)$ $N = 10$ distance between g_2

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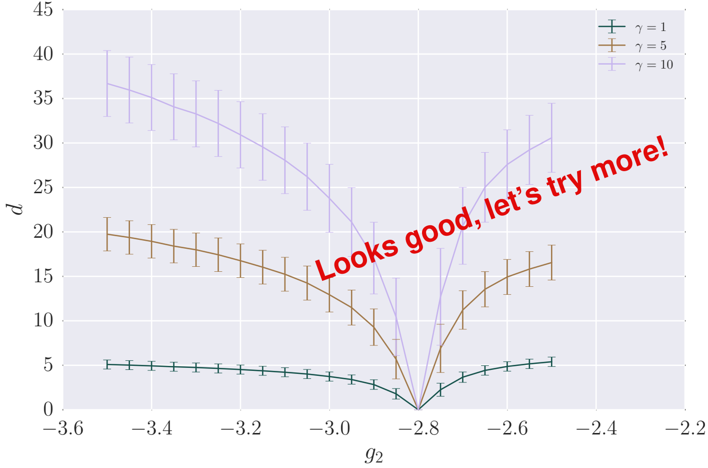

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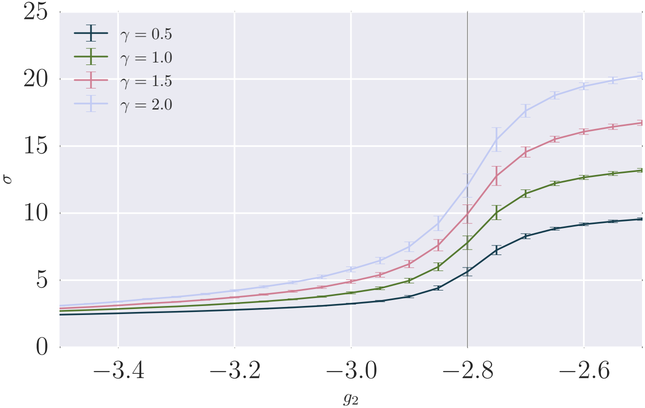
Distance from the fuzzy sphere



Type $(2, 0)$

Distance from the fuzzy sphere

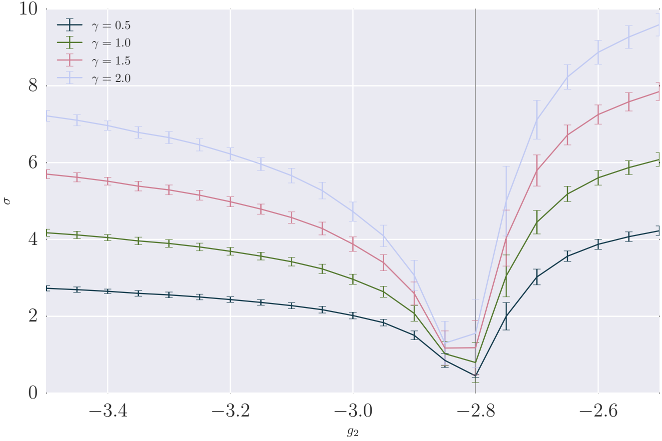
What is the difference?

Volume differences are also measured, we want 'shape'-differences.

- $\lambda_{max} = 1$ ➔ equal Planck lengths
- $R = 1$ ➔ equal volumes

R is hard to define, so we check the other option.

Distance from the fuzzy sphere



Type $(2, 0)$

The plan

- Spectral triples
- Fuzzy spaces
- Spectral observables
- Visualizing Truncated triples

Truncating a spectral triple

Truncate D

Replace the infinite D by a $n \times n$ matrix

$$D \to P_n D P_n$$

with P_n a projector on the n smallest eigenvalues.

- Mathematically a truncation of a spectral triple leads to an operator system, which has recently gathered much interest.
  (A. Connes, W.D. van Suijlekom, Commun.Math.Phys.(2020), W.D. van Suijlekom, arXiv:2005.08544)
- explored one ensemble of truncations of spectral triples
  (LG, A. Stern, J.Math.Phys. 61, 033507 (2020))
- Today: How can we recover distances from a truncated spectral triple and visualize it.
  (LG, A. Stern arXiv:1912.09227 (accepted by J.Geo.Phys.))

Non-commutative distance

Distance measure in non-commutative geometry

(A. Connes, Noncommutative Geometry. (Academic Press, 1994))

$$d(\omega_1, \omega_2) = \sup_{a \in \mathcal{A}} \{ |\omega_1(a) - \omega_2(a)| : ||[D, a]|| \leq 1 \}$$

Example:

Calculate distance between points x, y from function f

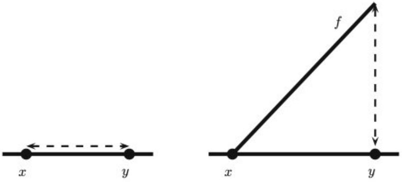


figure from (W.D. van Suijlekom "Noncommutative Geometry and Particle Physics" Springer (2015))

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Idea:

If we can calculate this numerically we can plot our geometry!
Maybe we can see a difference between the two Dirac operators?

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Questions:

- What are algebra elements a?
  Dirac we find has same eigenstates as sphere, can use truncated spherical harmonics as basis for $P_n C^\infty(S^2) P_n$
- Which states ω?
  Use localized states, inspired by

  (L. Schneiderbauer, H. Steinacker 2016, J.Phys. A49 285301)

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Aim:

Use the Kantorovich-Rubinstein distance between states of small dispersion to build a picture of M.

How do we define states?

Localized states

We use the dispersion and the embedding maps Y_i from the Heisenberg relations

$$\delta(\omega_k) = \sum_i \langle\omega|Y_i^2|\omega\rangle - \langle\omega|Y_i|\omega\rangle^2 + \sum_{j<k} \frac{c}{\delta(\omega_j, \omega_k)}$$

Now find a set of coherent states ω that minimizes this and plug them into distance equation. The repulsive potential is to ensure even distribution of points.

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Limitations:

We are using the Y_i to define the states and calculate a distance approximation δ. This is a good choice for geometries close to the sphere but can be abritrarily bad otherwise.

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We use the dispersion and the embedding maps Y_i from the Heisenberg relations

$$\delta(\omega_k) = \sum_i \langle\omega|Y_i^2|\omega\rangle - \langle\omega|Y_i|\omega\rangle^2 + \sum_{j<k} \frac{c}{\delta(\omega_j, \omega_k)}$$

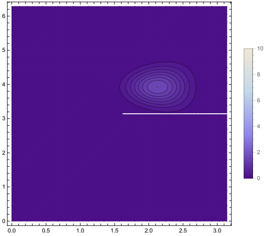
Now find a set of coherent states ω that minimizes this and plug them into distance equation. The repulsive potential is to ensure even distribution of points.

Advantage:

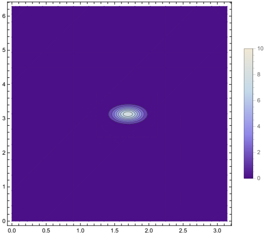
We can use it to plot the states and the generated geometry using the Y_i as embedding coordinates, for illustration purposes.

How does the state size change with the cutoff?

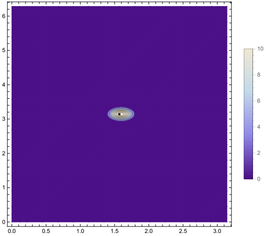
State for $\Lambda = 4$



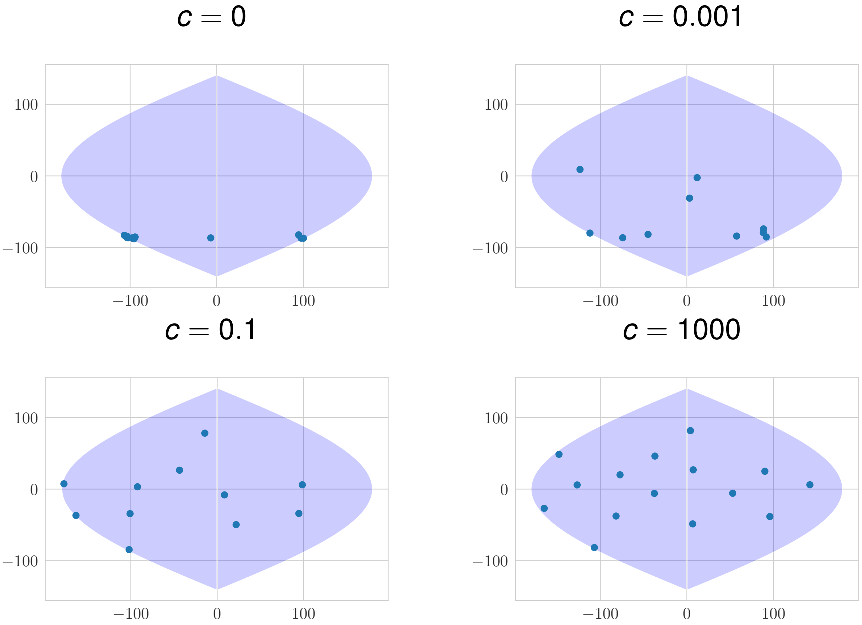
State for $\Lambda = 10$



State for $\Lambda = 16$



What effect does the repulsive potential have

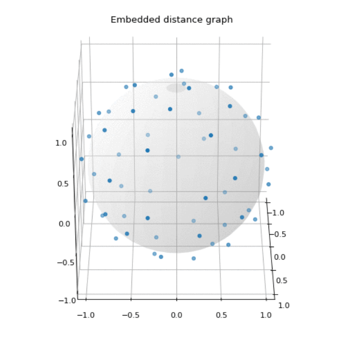


The algorithm for state generation

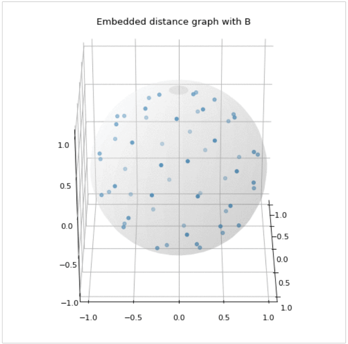
Find a vector v_0 (globally) minimizing δ. Set $V = \{v_0\}$.
while $\sqrt{\delta(v)} + \sqrt{\delta(w)} \leq \alpha d(v, w)$ for $v \neq w \in V$, **do**
    Find a vector w (locally) minimizing $e(w; V)$.
    Append w to V.
    for $v \in V$, **do**
        Set $d(v, w) = \min\{|\langle v, av \rangle - \langle w, aw \rangle| : |[D, a]| \leq 1\}$.
    end for
end while

A picture of geometry

The truncated sphere at size 60



The analytic solution at size 60



- generate states for a $n \times n$ matrix & calculate pairwise distances
- use graph embedding algorithm to find a locally isometric embedding
- wonder why the analytic solution is smaller

Summary

Todays story:

- Exploring NCG using computer simulations
- simulations in fuzzy spaces
- truncated NCGs as basis for simulations
- first numerical tests of one sided Heisenberg relation and Connes distance function

Immediate follow up:

- What is the difference between the two geometries?
- More simulations:
  - two-sided Heisenberg equation
  - path integral using Heisenberg equation as constraint
- More efficient imaging
  $\Rightarrow$ Use imaging on more states

Thanks for inviting me to talk,

and I hope I can visit Helsinki some time in the future!

Contact:
Email: lisa.glaser@univie.ac.at
Twitter: @GravityWithHat

Full axioms of non-commutative geometry

A finite real spectral triple consists of

1. The n-th characteristic value of the resolvent of D is $O(n^{-\frac{1}{p}})$.
2. $[[D, a], b] = 0 \forall a, b \in \mathcal{A}$
3. For any $a \in \mathcal{A}$ both a and $[D, a]$ belong to the domain of δ^m , for any integer m where δ is the derivation: $\delta(T) = [|D|, T]$.
4. There exists a Hochschild cycle $c \in Z_p(\mathcal{A}, \mathcal{A})$ such that $\pi_D(c) = 1$ for p odd, while for p even, $\pi_D(c) = \gamma$ is a $\mathbb{Z}/2$ grading.
5. Viewed as an $\mathcal{A}$-module the space $\mathcal{H}_\infty = \bigcap DomD^m$ is finite and projective. Moreover the following equality defines a hermitian structure $(|)$ on this module: $\langle \xi, a\eta \rangle = ⨍ a(\xi|\eta)|D|^{-p}, \forall a \in \mathcal{A}, \forall \xi, \eta \in \mathcal{H}_\infty$

(as summarized in A. Connes, Commun.Math.Phys. 182, 155-176 (1996))

Back to the abridged version.

States are points

Aim:

Use the Kantorovich-Rubinstein distance between states of small dispersion to build a picture of M.

Let $(C^\infty(M), H, D)$ be a commutative spectral triple equipped with a (not necessarily Riemannian) embedding $\iota : M \to \mathbb{R}^N$ (whose components are viewed as a set Y_i of generators of $C^\infty(M)$), and define the dispersion δ of a state ω to equal $\sum_i \omega(Y_i^2) - \omega(Y_i)^2$.

Lemma

There exists a map $b : S(A) \to M$ such that

$$|d(\omega_1, \omega_2) - d(b(x_1), b(x_2))| = O(\sqrt{\delta(\omega_1)} + \sqrt{\delta(\omega_2)}),$$

as $\delta(\omega_i) \to 0$, uniformly in ω_i.

Points are states

- Is there a picture of M inside $P_\Lambda H$, if Λ is large enough?

'Localization' map $\phi_\Lambda : M \to S(B(H))$ that factors through $P_\Lambda H$ and maps points $x \in M$ to corresponding vector states $\phi_\Lambda(x)$, such that $\phi_\Lambda(x)$ has asymptotically vanishing dispersion and $d(\phi_\Lambda(x), \phi_\Lambda(y))$ eventually equals $d(x, y)$.

Lemma

There exists a map $\phi : M \times \mathbb{R}_+ \to S(B(H))$, $(x, \Lambda) \mapsto \phi_\Lambda(x)$, such that

- *For all Λ, ϕ_Λ factors through a map $L_\Lambda : M \to P_\Lambda H$.*
- *The dispersion $\sum_i \phi_\Lambda(x)(Y_i^2) - \phi_\Lambda(x)(Y_i)^2$ is $O(\Lambda^{-2}\log\Lambda)$ as $\Lambda \to \infty$.*
- *For all $x, y \in M$, $|d(\phi_\Lambda(x), \phi_\Lambda(y)) - d(x, y)| = O(\Lambda^{-1}(\log\Lambda)^{\frac{1}{2}})$.*

Larger cutoff ⇒ more points

We could alternatively have phrased the third point in the lemma as follows: the maps ϕ_Λ and b (from 1) are asymptotically inverse to each other in the sense that $d(x, (b \circ \phi_\Lambda)(x)) = O(\Lambda^{-1})$ and $d((\phi_\Lambda \circ b)(\omega), \omega) \leq \sqrt{\delta(\omega)} + O(\Lambda^{-2})$.
In particular the previous lemma tells us how to scale the number of generated states with Λ:

Corollary

A sequence of equidistributed subsets $\{V_n\}_n$ of M, in the sense that $\min d|_{V_n \times V_n \setminus \Delta} = \Theta(|V_n|^{-1/n})$, will satisfy

$$\sup_{x,y \in V_n} \frac{|d(x,y) - d(\phi_{\Lambda_n}(x), \phi_{\Lambda_n}(y))|}{d(x,y)} = O(1)$$

as $\Lambda \to \infty$, whenever $|V_n| = \Theta(\operatorname{rank} P_{\Lambda_n})$.