# Primordial gravitational waves (and $N_{\mathrm{eff}}$ bounds revisited)

**Subodh Patil**
**Leiden University**
**University of Helsinki seminar**
**27/03/2024**

# Most people wouldn't come to this talk:

## An Étude on the Regularization and Renormalization of Divergences in Primordial Observables

Anna Negro[*] and Subodh P. Patil[†]

*Instituut-Lorentz for Theoretical Physics,*
*Leiden University, 2333 CA Leiden, The Netherlands*

(Dated: February 20, 2024)

Many cosmological observables of interest derive from primordial vacuum fluctuations evolved to late times. These observables represent statistical draws from some underlying quantum or statistical field theoretic framework where infinities arise and require regularization. After subtracting divergences, renormalization conditions must be imposed by measurements or observations at some scale, mindful of scheme and background dependence. We review this process on backgrounds that transition from finite duration inflation to radiation domination, and show how in spite of the ubiquity of scaleless integrals, UV divergences can still be meaningfully extracted from quantities that nominally vanish when dimensionally

arXiv:2402.10008; La Rivista del Nuovo Cimento, *in press*

# Or come to this talk either…

## Hadamard Regularization of the Graviton Stress Tensor


Anna Negro* and Subodh P. Patil†

*Instituut-Lorentz for Theoretical Physics,*
*Leiden University, 2333 CA Leiden, The Netherlands*



(Dated: March 26, 2024)



We present the details for the covariant renormalization of the stress tensor for vacuum tensor perturbations at the level of the effective action, adopting Hadamard regularization techniques to isolate short distance divergences and gauge fixing via the Faddeev-Popov procedure. The subsequently derived renormalized stress tensor can be related to more familiar forms reliant upon an averaging prescription, such as the Isaacson or Misner-Thorne-Wheeler forms. The latter, however, are premised on a prior scale separation (beyond which the averaging is invoked) and therefore unsuited for the purposes of renormalization. This can lead to potentially unphysical conclusions when taken as a starting point for the computation of any observable that needs regularization, such as the energy density associated to a stochastic background. Any averaging prescription, if needed, should only be invoked at the end of the renormalization procedure. The latter necessarily involves the imposition of renormalization conditions via a physical measurement at some fixed scale, which we retrace for primordial gravitational waves sourced from vacuum fluctuations through direct or indirect observation.



arXiv:2403.16806

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**Hadamard Regularization of the Graviton Stress Tensor**

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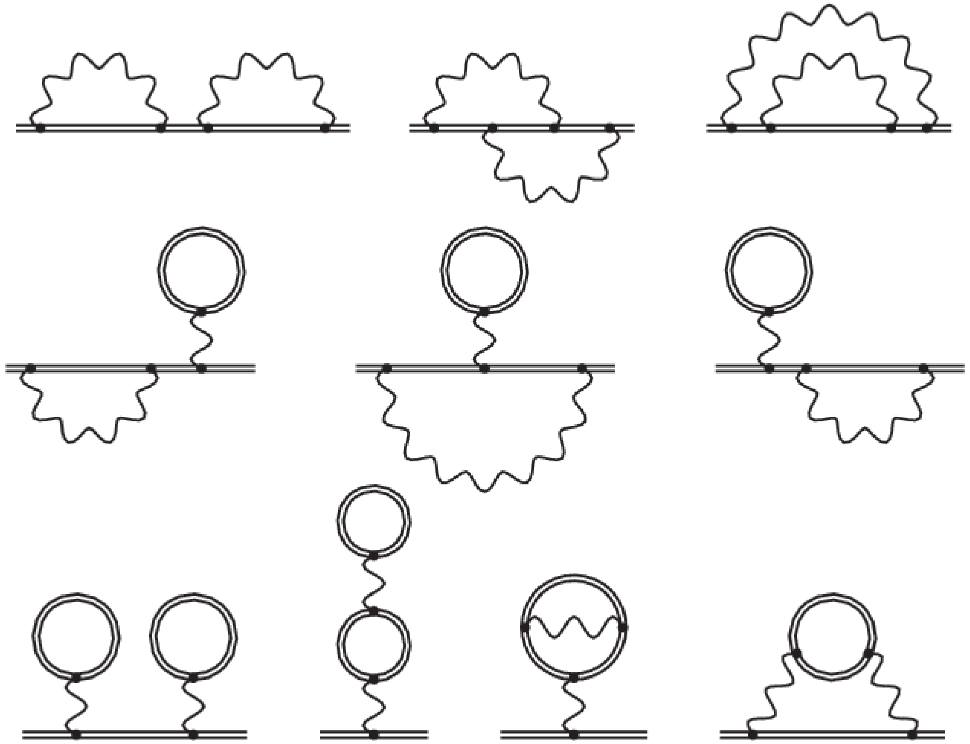
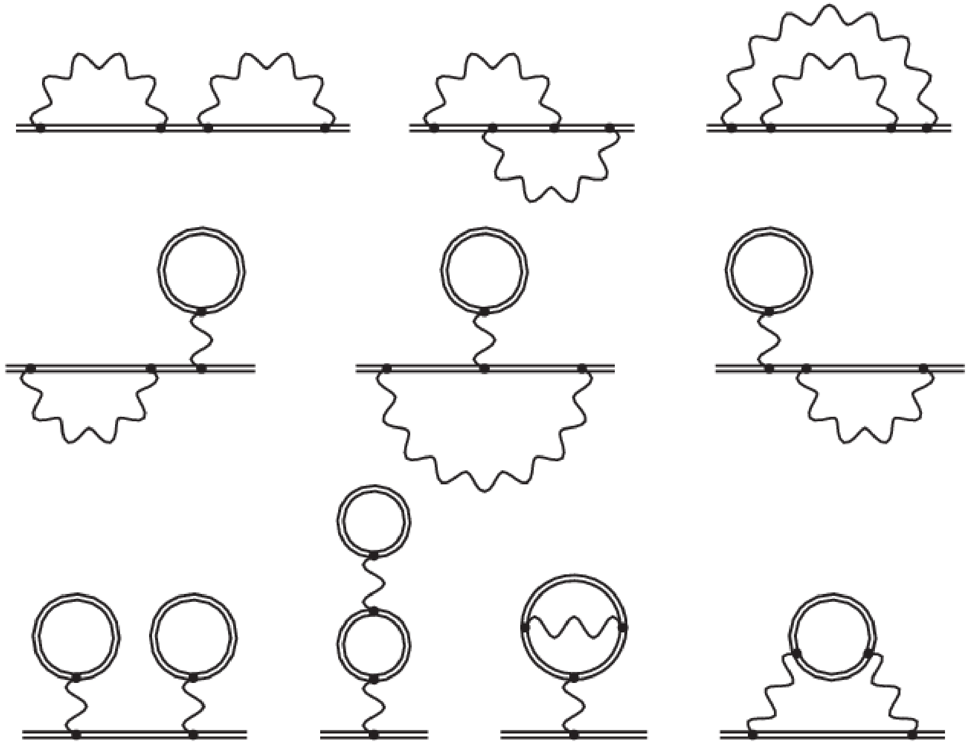
but it's the talk you're more or less going to get…

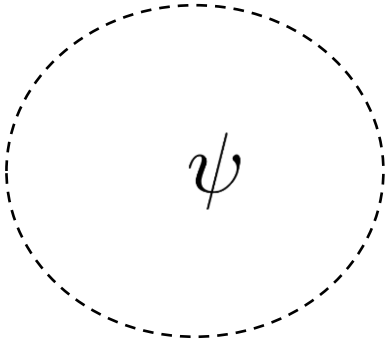
# What is a `quantum correction’?



# Do we get to calculate corrections to `classical' quantities?



# Say I calculate a bubble diagram on flat space:

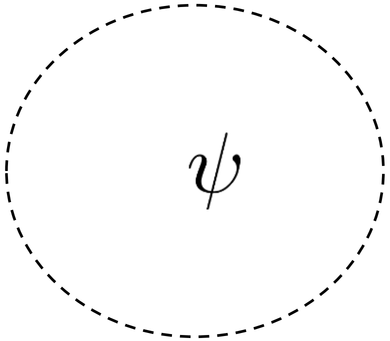



$$W = \tfrac{1}{2}\log\left[\det(-\Box + m_\psi^2)\right]$$

$$\begin{aligned} W &= \frac{1}{2}\int_{\frac{1}{\Lambda^2}}^{\infty} \frac{\mathrm{d}s}{s} \int d^4x \left\langle x \middle| e^{-s[-\Box + m_\psi^2]} \middle| x \right\rangle \\ &= \frac{1}{2}\int_{\frac{1}{\Lambda^2}}^{\infty} \frac{\mathrm{d}s}{s} \int \mathrm{d}^4x G(x,x;s) = \frac{1}{32\pi^2}\int \mathrm{d}^4x \int_{\frac{1}{\Lambda^2}}^{\infty} \frac{\mathrm{d}s}{s^3} e^{-m_\psi^2 s} \end{aligned}$$

$$\frac{1}{32\pi^2}\int_{\frac{1}{\Lambda^2}}^{\infty} \frac{\mathrm{d}s}{s^3} e^{-m_\psi^2 s} = \frac{1}{64\pi^2}\left[\Lambda^4 - m_\psi^4 \ln\left(m_\psi^2/\Lambda^2\right)\right]$$

# Does this mean Minkowski gets corrected to dS?

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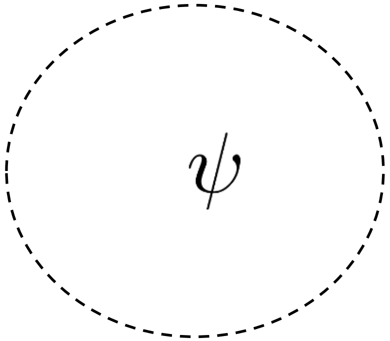
# Obviously not – regularization subtracts infinities, renormalization conditions fix finite parts…

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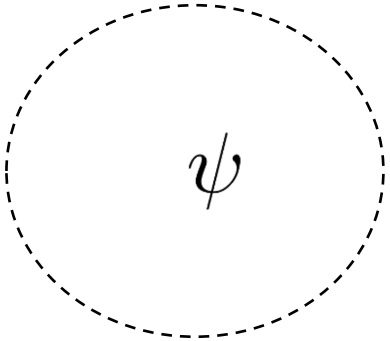
# Once you've fixed renormalization conditions, all one gets to calculate is how things change with scale…

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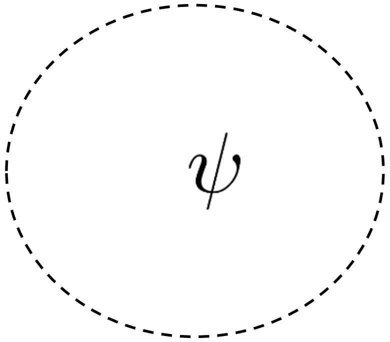
# The cosmological constant problem is a statement about the instability of flat space as one changes scale…

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# Sometimes the idea of a quantum `correction' is itself a category error – the only reality is the fully dressed one...

Consider the following Lagrangian in 1+1 D:

$$\mathcal{L}(x) = \frac{1}{2}\partial_\mu \vartheta(x) \partial^\mu \vartheta(x)$$

Can quantize as usual – plane wave basis states etc...

# Sometimes the idea of a quantum `correction’ is itself a category error – the only reality is the fully dressed one...

But what about this one?

$$\mathcal{L}(x) = \frac{1}{2}\partial_\mu \vartheta(x)\partial^\mu \vartheta(x) + \frac{\alpha}{\beta^2}(\cos\beta\vartheta(x) - 1)$$

Hard to quantize as usual around plane wave basis…

# Sometimes the idea of a quantum `correction' is itself a category error – the only reality is the fully dressed one...

But what about this one?

$$\mathcal{L}(x) = \tfrac{1}{2}\partial_\mu \vartheta(x)\partial^\mu \vartheta(x) + \tfrac{\alpha}{\beta^2}(\cos\beta\vartheta(x) - 1)$$

Start to notice some funny things: solitons with like topological charges repel, equal and opposite annihilate…

# Reminder: (Classical) Lagrangians are not physical

$$\mathcal{L}(x) = \tfrac{1}{2}\partial_\mu \vartheta(x)\partial^\mu \vartheta(x) + \tfrac{\alpha}{\beta^2}(\cos\beta\vartheta(x) - 1)$$

$$\begin{aligned}\mathcal{L}(x) = &\bar{\psi}(x)\left(i\gamma^\mu \partial_\mu - m\right)\psi(x) \\ &- \frac{1}{2}g\bar{\psi}(x)\gamma^\mu\psi(x)\bar{\psi}(x)\gamma_\mu\psi(x)\end{aligned}$$

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**Skyrme (1964): Sine-Gordon solitons very much are the Thirring fermions in disguise…**

# Reminder: (Classical) Lagrangians are not physical

**Proved at operator and GF level** (Mandelstam, Coleman 1975)

$$\mathcal{L}(x) = \tfrac{1}{2}\partial_\mu \vartheta(x) \partial^\mu \vartheta(x) + \tfrac{\alpha}{\beta^2}(\cos\beta\vartheta(x) - 1)$$

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$$Zm\bar{\psi}(x)\frac{1 \mp \gamma^5}{2}\psi(x) = -\frac{\alpha}{\beta^2} e^{\pm i\vartheta(x)}$$

$$\frac{4\pi}{\beta^2} = 1 + \frac{g}{\pi}$$

# Loop corrections and anomalous dimensions

- In 1+1: $[\bar{\psi}\psi] = 1$ ,$[\cos(\beta\vartheta)] = 0$ get mapped to each other, so the latter has anomalous dimension of one (cf. compositeness).

- Anybody doing scattering experiments with Thirring Fermions sees only the fully dressed reality.

# Outline

- Regularizing divergences – making sense of hard cutoffs, dim reg, point splitting, UV vs IR etc.

- UV divergences can still be extracted from scaleless integrals in dim reg.

- Finite duration inflation – distinguishing UV and IR scales from unknown completion of theory/ observables from beginning and end of inflation.

- Aside – various IR divergences are cured in finite duration inflation.

- Vacuum stress tensor of tensor perturbations – need to go beyond Isaacson.

- Energy density of tensor perturbations – observable, or shifted tadpole condition?

- Relation to extracting Neff bounds + consequences.

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# Divergences are not an academic issue!

- All physical observables: momentum or energy transfer between propagating degrees of freedom and some sort of detector or tracer.

- Primordial correlation functions are a useful calculational intermediary between observations and underlying effective description, but not directly observable.

- Need to be acted on with derivatives and convolved with transfer functions.

- Physical observations are made at some point in space or time, sample coincident limit of field bilinears or higher point functions.

- Observations cannot be made sense of unless divergences in the coincident limit subtracted and renormalized.

“Just because something is infinite, doesn’t mean it is zero” – unattributed quote

# Divergences are not that bad…

- Consider the energy momentum tensor of a minimally coupled, non-interacting test scalar field:

$$T^{\mu}_{\nu} = \partial^{\mu}\phi\partial_{\nu}\phi - \frac{1}{2}\delta^{\mu}_{\nu}\left(g^{\lambda\beta}\partial_{\lambda}\phi\partial_{\beta}\phi + m^2\phi^2\right)$$

$$-T^0_0 := \rho = \frac{\phi'^2}{2a^2} + \frac{(\nabla\phi)^2}{2a^2} + \frac{m^2}{2}\phi^2$$

$$\rho(\tau; x, y) := \frac{1}{2a^2}\left[\phi'(\tau, x)\phi'(\tau, y) + \nabla_x\phi(\tau, x)\cdot\nabla_y\phi(\tau, y) + m^2a^2\phi(\tau, x)\phi(\tau, y)\right]$$

- Start with the two point function and proceed from there:

$$\langle\phi(\tau, x)\phi(\tau, y)\rangle = \int \frac{d^3k}{4\pi}\frac{\mathcal{P}_{\phi}(\tau, k)}{k^3}e^{ik\cdot(x-y)},$$
$$= \int \frac{dk}{k}\mathcal{P}_{\phi}(\tau, k)\frac{\sin(kr)}{kr},$$

# Divergences are not that bad…

- The coincident limit for a massless test scalar on an inflating background

$$\lim_{\tau \to 0} \langle \phi(\tau, x)\phi(\tau, x)\rangle \sim \left(\frac{H_0}{2\pi}\right)^2 \int_0^\infty \frac{dk}{k}\left(\frac{k}{H_0}\right)^{n_s-1}\left[1+\left(\frac{k}{aH_0}\right)^2 \frac{a^{n_s-1}}{2-n_s} + \ldots\right]$$

- de Sitter is particularly simple –

$$\lim_{\tau \to 0} \langle \phi(\tau, x)\phi(\tau, x)\rangle = \left(\frac{H_0}{2\pi}\right)^2 \int_0^\infty \frac{dk}{k}\left[1+\left(\frac{k}{aH_0}\right)^2\right]$$

- All integrals are scaleless… in mass independent schemes, the above would vanish. Considering massive but sufficiently light fields on dS shows the same:

$$\lim_{\tau \to 0} \langle \phi(\tau, x)\phi(\tau, x)\rangle = \frac{H_0^2}{8\pi^3} 2^{2\nu_m}\Gamma^2(\nu_m) \int_0^\infty \frac{dk}{k}\left[\left(\frac{k}{aH_0}\right)^{3-2\nu_m} + \frac{1}{2(\nu_m-1)}\left(\frac{k}{aH_0}\right)^{5-2\nu_m}\right]$$

# But just because something is zero doesn't mean it isn't infinite!

- Lesson from matching calculations in NR QED/QCD: Can factor scaleless integrals into a scaleful ones:

$$\int_0^\infty \frac{dk}{k} = \int_0^\infty \frac{k^3 dk}{k^4} = \frac{1}{2\pi^2}\int_{-\infty}^\infty \frac{d^4k}{k^4}$$

$$I_D(m^2) = \int \frac{d^Dk}{(2\pi)^D}\frac{k^{2A}}{(k^2+m^2)^B} \qquad I_D(m^2) = \frac{\Gamma\left(A+\frac{D}{2}\right)\Gamma\left(B-A-\frac{D}{2}\right)}{(4\pi)^{D/2}\Gamma\left(\frac{D}{2}\right)\Gamma(B)}\left(m^2\right)^{A-B+D/2}$$

$$\lim_{\tau\to 0}\langle\phi(\tau,x)\phi(\tau,x)\rangle = \left(\frac{H}{2\pi}\right)^2\int_0^\infty \frac{dk}{k}.$$

$$\lim_{\tau\to 0}\langle\phi(\tau,x)\phi(\tau,x)\rangle = \frac{H^2}{8\pi^4}\int_{-\infty}^\infty \frac{d^4k}{k^4} = \frac{H^2}{8\pi^4}\int_{-\infty}^\infty d^4k\left[\frac{1}{k^2(k^2+m^2)} + \frac{m^2}{k^4(k^2+m^2)}\right]$$

# But just because something is zero doesn't mean it isn't infinite!

- Integrals give equal and opposite contributions:

$$\lim_{\tau\to 0}\langle\phi(\tau,x)\phi(\tau,x)\rangle = \frac{H^2}{8\pi^4}\int_{-\infty}^{\infty}\frac{d^4k}{k^4} = \frac{H^2}{8\pi^4}\int_{-\infty}^{\infty} d^4k\left[\frac{1}{k^2(k^2+m^2)}+\frac{m^2}{k^4(k^2+m^2)}\right]$$

$$\pm\frac{H^2}{4\pi^2}\left[\frac{1}{\delta}-\frac{1}{2}\left(\log\frac{m^2}{4\pi\mu^2}+\gamma_E-1\right)\right]$$

$$\lim_{\tau\to 0}\langle\phi(\tau,x)\phi(\tau,x)\rangle = \left(\frac{H}{2\pi}\right)^2\left[\frac{1}{\delta_{\rm UV}}-\frac{1}{\delta_{\rm IR}}+\log\frac{\mu_{\rm UV}}{\mu_{\rm IR}}\right]$$

- UV divergences are unambiguous. IR divergences could mean any number of things…

# But just because something is zero doesn't mean it isn't infinite!

- Let's reconsider putting a hard cutoff in physical momenta:

$$\lim_{\tau\to 0}\langle\phi(\tau,x)\phi(\tau,x)\rangle = \left(\frac{H_0}{2\pi}\right)^2\int_0^\infty \frac{dk}{k}\left[1+\left(\frac{k}{aH_0}\right)^2\right]$$

$$\lim_{\tau\to 0}\langle\phi(\tau,x)\phi(\tau,x)\rangle = \left(\frac{H_0}{2\pi}\right)^2\left\{\int_{a\Lambda_{\rm IR}}^{a\Lambda_{\rm UV}} \frac{dk}{k}\left[1+\left(\frac{k}{aH_0}\right)^2\right] - \int_{a\mu}^{a\Lambda_{\rm UV}} \frac{dk}{k}\left[1+\left(\frac{k}{aH_0}\right)^2\right]\right\}$$

$$= \left(\frac{H_0}{2\pi}\right)^2\left\{\log\left(\frac{\mu}{\Lambda_{\rm IR}}\right)+\frac{1}{2H_0^2}\left(\mu^2-\Lambda_{\rm IR}^2\right)\right\}, \qquad (n_s=1)$$

- Appearance of IR divergence means we have yet to arrive at a physical observable…

- UV divergence subtracted by a cosmological constant counterterm:

$$\text{c.t.} = \left(\frac{H_0}{2\pi}\right)^2\left\{\log\left(\frac{\mu}{\Lambda_{\rm UV}}\right)+\frac{1}{2H_0^2}\left(\mu^2-\Lambda_{\rm UV}^2\right)\right\}$$

# But just because something is zero doesn't mean it isn't infinite!

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$$= \left(\frac{H_0}{2\pi}\right)^2 \left\{\log\left(\frac{\mu}{\Lambda_{\rm IR}}\right) + \frac{1}{2H_0^2}\left(\mu^2 - \Lambda_{\rm IR}^2\right)\right\}, \qquad (n_s = 1)$$

- Appearance of IR divergence means we have yet to arrive at a physical observable…

- Matches UV counterterm identified in dim reg:

$$\text{c.t.} = \left(\frac{H_0}{2\pi}\right)^2 \left\{\log\left(\frac{\mu}{H_0}\right) - \frac{1}{\delta_{\rm UV}} + \frac{1}{2}\left(\gamma_E - 1 - \log 4\pi\right)\right\}$$

# UV log divergences agree in all schemes

- Massless fields on quasi dS:

$$\text{c.t.} = \lim_{\epsilon\to 0} \frac{a^{-4\epsilon}\Lambda_{\rm UV}^2}{4\pi^2}\left\{\frac{1}{2\epsilon} - \log\frac{\Lambda_{\rm UV}}{H_0} + ...\right\}$$

$$\begin{aligned}\text{c.t.} &= \lim_{\epsilon\to 0} -\frac{a^{2(n_s-1)}}{2-n_s}\left(\frac{H_0}{2\pi}\right)^2 \frac{\mu^2}{H_0^2}\left\{\frac{1}{n_s-1} + \log\frac{\mu}{H_0}\right\} \\ &= \lim_{\epsilon\to 0} \frac{a^{-4\epsilon}\mu^2}{4\pi^2}\left\{\frac{1}{2\epsilon} - \log\frac{\mu}{H_0} + ...\right\}\end{aligned}$$

- Time dependent counterterm via: $\text{c.t.} \subset \int d^4x\sqrt{-g}\left[c_1(\mu)R^2 + c_2(\mu)R_{\mu\nu}R^{\mu\nu}\right]$

- Massive fields on dS

$$\text{c.t.} = \frac{H_0^2}{8\pi^2}\frac{2^{2\nu_m}\Gamma^2(\nu_m)}{\pi}\left[\frac{3H_0^2}{m^2} + \log\frac{\Lambda_{\rm UV}}{H_0} + ...\right] \quad \text{(physical cutoff)}$$

$$\text{c.t.} = \frac{H_0^2}{8\pi^2}\frac{2^{2\nu_m}\Gamma^2(\nu_m)}{\pi}\left[\frac{3H_0^2}{m^2} + \log\frac{\mu}{H_0} + ...\right] \quad \text{(dim reg)}$$

# But not all schemes are equal…

- If you cannot identify a consistent counter term within your regularization scheme, you cannot be guaranteed to be calculating physical quantities.

- C.f. energy momentum tensor of test scalar field on dS:

$$\rho = \frac{H^4}{8\pi^2}\int_0^\infty \frac{dk}{k}\left[\left(\frac{k}{aH}\right)^2 + 2\left(\frac{k}{aH}\right)^4\right]$$

$$p = \frac{H^4}{8\pi^2}\int_0^\infty \frac{dk}{k}\left[-\frac{1}{3}\left(\frac{k}{aH}\right)^2 + 2\left(\frac{k}{aH}\right)^4\right]$$

- Physical cutoffs requires counterterm that cannot be constructed from geometric invariants. In dim reg, however, c.t. $\propto g_{\mu\nu}$, i.e. renormalization of c.c.

- (coefficients of log divergences still agree though :)

# Outline

- Regularizing divergences – making sense of hard cutoffs, dim reg, point splitting, UV vs IR etc.
- UV divergences can still be extracted from scaleless integrals in dim reg.
- Finite duration inflation – distinguishing UV and IR scales from unknown completion of theory/ observables from beginning and end of inflation.
- Aside – various IR divergences are cured in finite duration inflation.
- Vacuum stress tensor of tensor perturbations – need to go beyond Isaacson.
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# Finite duration inflation

- Motivation for this and the next part of the talk – a typical expression one might find in the literature:

$$\rho_{\rm GW} \simeq \frac{A_{\rm t}}{32\pi G_N}\left(\frac{k_{\rm UV}}{k_*}\right)^{n_{\rm t}} \frac{1}{2n_{\rm t}}\frac{1}{a^4} \propto \frac{1}{a^4}\left[\frac{1}{n_{\rm t}} + \log\frac{k_{\rm UV}}{k_*}\right]$$

- UV and IR scales from unknown completion must be distinguished form UV and IR scales corresponding to beginning and end of inflation.

- Observables cannot depend on the former, but can certainly depend on the latter.

- Bonus observation – certain IR divergences can be shown to be an artefact of a past infinite dS approximation.

# Finite duration inflation

- Consider a cosmology that transitions into and out of inflation to radiation domination:

$$
\begin{aligned}
a(\tau) &= a_{\rm R}\left(2 - \frac{\tau}{\tau_{\rm I}}\right) e^{-\mathcal{N}_{\rm tot}} & \tau < \tau_{\rm I} \\
&= a_{\rm R}\left(\frac{\tau_{\rm I}}{\tau}\right) e^{-\mathcal{N}_{\rm tot}} & \tau_{\rm I} < \tau < \tau_{\rm R} \\
&= a_{\rm R}\left(2 - \frac{\tau}{\tau_{\rm R}}\right) & \tau_{\rm R} < \tau
\end{aligned}
$$

$$
\mathcal{N}_{\rm tot} = \log(a_{\rm R}/a_{\rm I}) = \log(\tau_{\rm I}/\tau_{\rm R})
$$

$$
H = -\frac{1}{a_{\rm R}\tau_{\rm R}}
$$

- Construct mode functions for a massless test scalar via matching from initial adiabatic vacuum state:

$$
\begin{aligned}
\alpha_k^{\rm I} &= \left(i - i\frac{a_{\rm I}^2 H^2}{2k^2} + \frac{a_{\rm I} H}{k}\right) \\
\beta_k^{\rm I} &= i\frac{a_{\rm I}^2 H^2}{2k^2} e^{2i\frac{k}{a_{\rm I} H}}.
\end{aligned}
$$

$$
\begin{aligned}
\alpha_k^{\rm R} &= \alpha_k^{\rm I}\left(-i + \frac{a_{\rm R} H}{k} + i\frac{a_{\rm R}^2 H^2}{2k^2}\right) + i\beta_k^{\rm I}\frac{a_{\rm R}^2 H^2}{2k^2} e^{-2i\frac{k}{a_{\rm R} H}} \\
\beta_k^{\rm R} &= -i\alpha_k^{\rm I}\frac{a_{\rm R}^2 H^2}{2k^2} e^{2i\frac{k}{a_{\rm R} H}} + \beta_k^{\rm I}\left(i + \frac{a_{\rm R} H}{k} - i\frac{a_{\rm R}^2 H^2}{2k^2}\right).
\end{aligned}
$$

# Finite duration inflation

- Consider the two coincident limit of the point function:

$$\lim_{x\to y}\langle\phi(\tau,x)\phi(\tau,y)\rangle = \frac{1}{2\pi^2a^2}\int_0^\infty \frac{dk}{k}\frac{k^2}{2}\left[1+2|\beta_k^{\rm R}|^2+\alpha_k^{\rm R}\beta_k^{\rm R*}e^{\frac{2ik}{a_{\rm R}H}\left(2-\frac{a}{a_{\rm R}}\right)}+\alpha_k^{\rm R*}\beta_k^{\rm R}e^{-\frac{2ik}{a_{\rm R}H}\left(2-\frac{a}{a_{\rm R}}\right)}\right]$$
$$= \frac{1}{2\pi^2a^2}\int_0^\infty \frac{dk}{k}\frac{k^2}{2}\left[1+2|\beta_k^{\rm R}|^2_{\rm power}+\{\rm osc\}\right]$$

- Straightforward to show that oscillatory terms do not contribute to UV divergences.

- On the other hand: $|\beta_k^{\rm R}|^2_{\rm power} = \frac{a_{\rm R}^4H^4+a_{\rm I}^4H^4}{4k^4}+\frac{a_{\rm I}^4a_{\rm R}^4H^8}{8k^8}$

- But oscillations freeze in the IR, cancel would be (aggravated) IR power divergences:

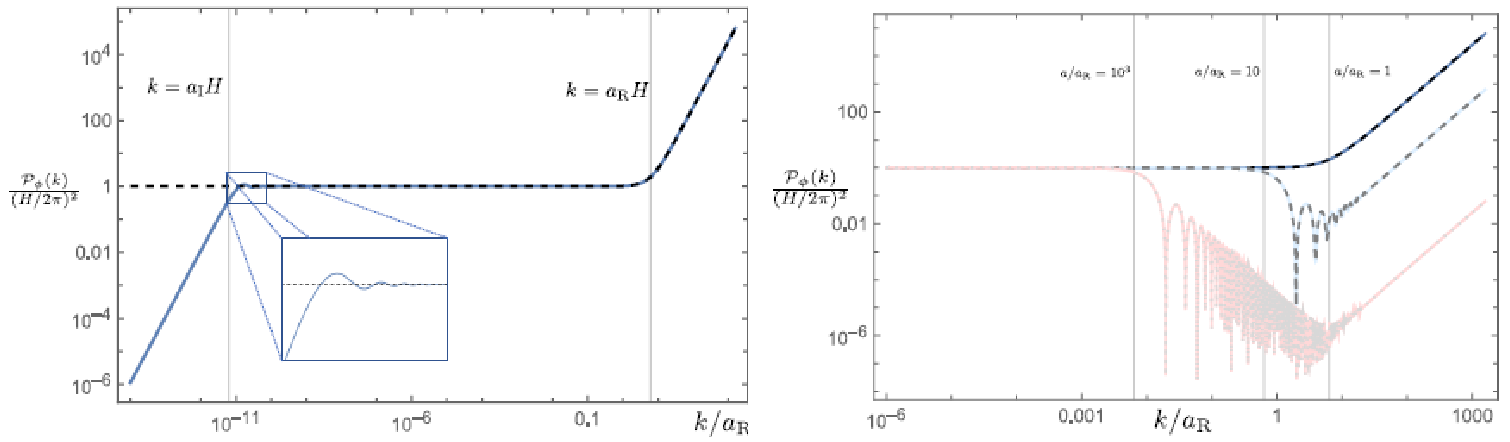
$$\lim_{k\to0}\{\rm osc\} = -\frac{a_{\rm R}^4H^4+a_{\rm I}^4H^4}{2k^4}-\frac{a_{\rm I}^4a_{\rm R}^4H^8}{4k^8}-1+\frac{(3a_{\rm I}^3a_{\rm R}+2a(a_{\rm R}^3-a_{\rm I}^3))^2}{9a_{\rm I}^2a_{\rm R}^6}$$

# Finite duration inflation

- Remaining UV divergence now *parameterized* by the beginning and end of inflation:

$$\lim_{x\to y}\langle\phi(\tau,x)\phi(\tau,y)\rangle_{\text{div}} = e^{-4\mathcal{N}_{\text{tot}}}\left(3+\frac{2a}{a_{\text{R}}}\left[e^{3\mathcal{N}_{\text{tot}}-1}\right]\right)^2\frac{1}{36\pi^2a^2}\int_0^\infty dk\,k$$

- IR divergences regulated by the physical scale corresponding to the start of inflation:




(a) Power spectra evaluated at reheating $a = a_{\text{R}}$, where $a_{\text{I}} = 10^{-12}a_{\text{R}}$ in units where $H$ is set to $2\pi$.

(b) Power spectra evaluated at different times during radiation domination.

# Finite duration inflation

- Can do the same with the power spectral energy density

$$\rho = \frac{1}{8\pi^2 a^4}\int_0^\infty \frac{dk}{k}\left[k^4\left(2+\frac{a_{\rm R}^4 H^2}{a^2k^2}\right)(1+2|\beta_k^{\rm R}|^2_{\rm power}) + \{\rm osc\}\right]$$
$$:= \int_0^\infty \frac{dk}{k}\left[\Omega^\phi_{\rm power}(k) + \Omega^\phi_{\rm osc}(k)\right]$$

- UV divergences: $\rho_{\rm div} = \frac{1}{8\pi^4 a^4}\int_{-\infty}^{\infty} d^4k\left(1+\frac{H^4 a_{\rm R}^4 + H^4 a_{\rm I}^4}{2k^4}\right) + \frac{1}{8\pi^4 a^6}\int_{-\infty}^{\infty} d^4k \frac{a_{\rm R}^4 H^2}{2k^2}$

- Coefficients of log divergence agree in all schemes:

$$\rho^{\log}_{\rm div} = \frac{H^4(1+e^{-4\mathcal{N}_{\rm tot}})}{8\pi^2(a/a_{\rm R})^4}\left[\frac{1}{\delta_{\rm UV}} + 1 - \gamma_E + \log\left(\frac{\mu}{H}\right)\right]$$

# Finite duration inflation

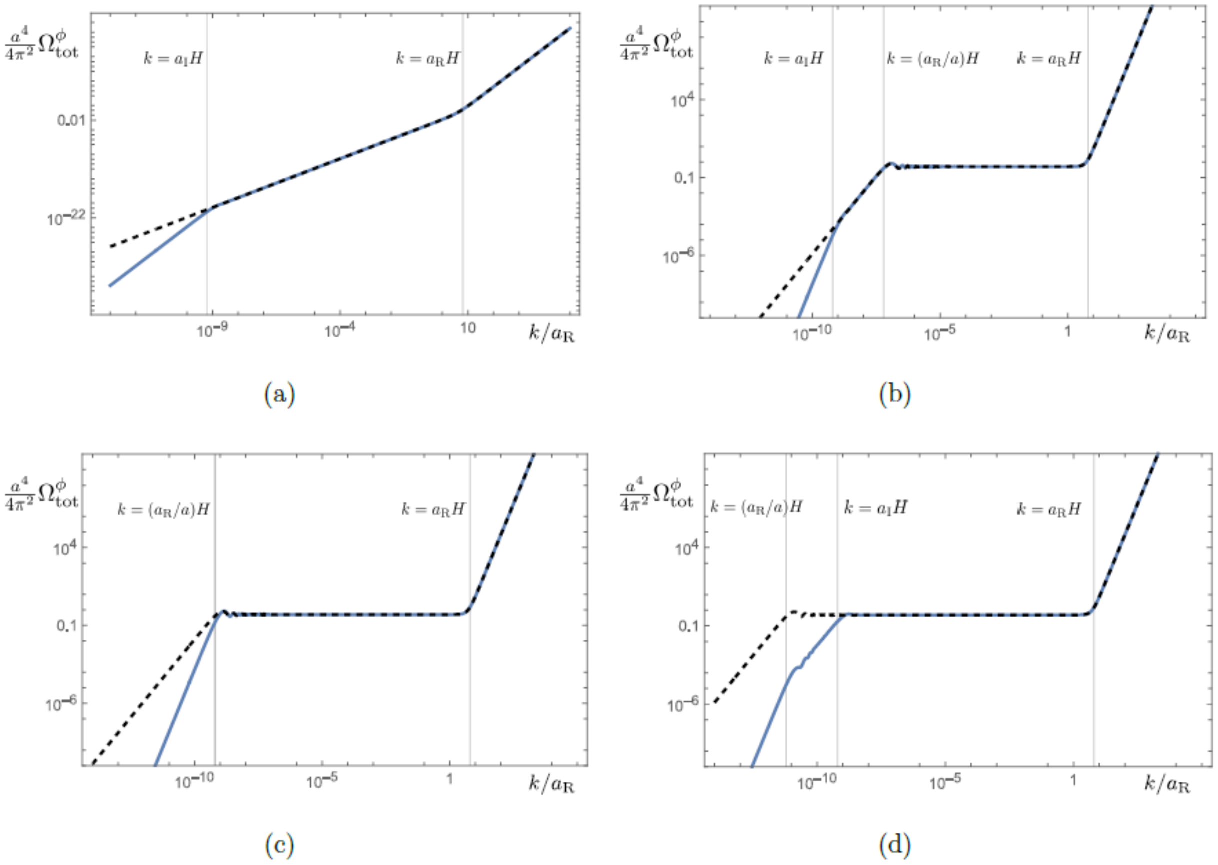



Figure 2: Power spectral energy density for a massless test scalar field comparing past infinite vs finite duration de Sitter inflation (dashed and bold lines, respectively) matched to a terminal stage of radiation domination. The panels are evaluated at subsequent times during radiation domination, with $a_{\rm I} = 10^{-10} a_{\rm R}$, and with $a/a_{\rm R} = 1, 10^{-}8, 10^{-10}$, and $10^{-12}$, respectively, in units where $H = 2\pi$.

# Outline

- Regularizing divergences – making sense of hard cutoffs, dim reg, point splitting, UV vs IR etc.
- UV divergences can still be extracted from scaleless integrals in dim reg.
- Finite duration inflation – distinguishing UV and IR scales from unknown completion of theory/ observables from beginning and end of inflation.
- Aside – various IR divergences are cured in finite duration inflation.
- Vacuum stress tensor of tensor perturbations – need to go beyond Isaacson.
- Energy density of tensor perturbations – observable, or shifted tadpole condition?
- Relation to  extracting Neff bounds + consequences.

# Isaacson stress tensor is not fit to purpose

- Assumes the Brill-Hartle averaging scheme – not valid for wavelengths comparable too background curvature:

$$\rho_{\rm gw}^{\rm Isc} = \frac{1}{32\pi a^2 G_N} \left\langle h'_{ij}(\tau, k) h'^{ij}(\tau, k) \right\rangle$$

- Instead we can retrace derivation without BH averaging, and undo approximations reliant on prior scale separation (cf. Maccallum-Taub w/o BH averaging).

$$\lim_{y\to x} \rho_{\rm gw}(\tau; x, y) = \frac{1}{8\pi a^2 G_N} \left\langle \frac{1}{8} h'_j{}^i h'^j_i - \frac{3}{8} \partial_k h_{ij} \partial^k h^{ij} - \frac{1}{2} h_{ij} \partial_k \partial^k h^{ij} + \frac{1}{4} \partial_k h_{ij} \partial^j h^{ik} + \mathcal{H} h_j{}^i h'^j_i \right\rangle .$$

- Perhaps this wasn't done before because people tripped over lack of positive definiteness of spectral density? This is a well known property of gravity (c.f. positive energy theorems).

# Isaacson stress tensor is not fit to purpose

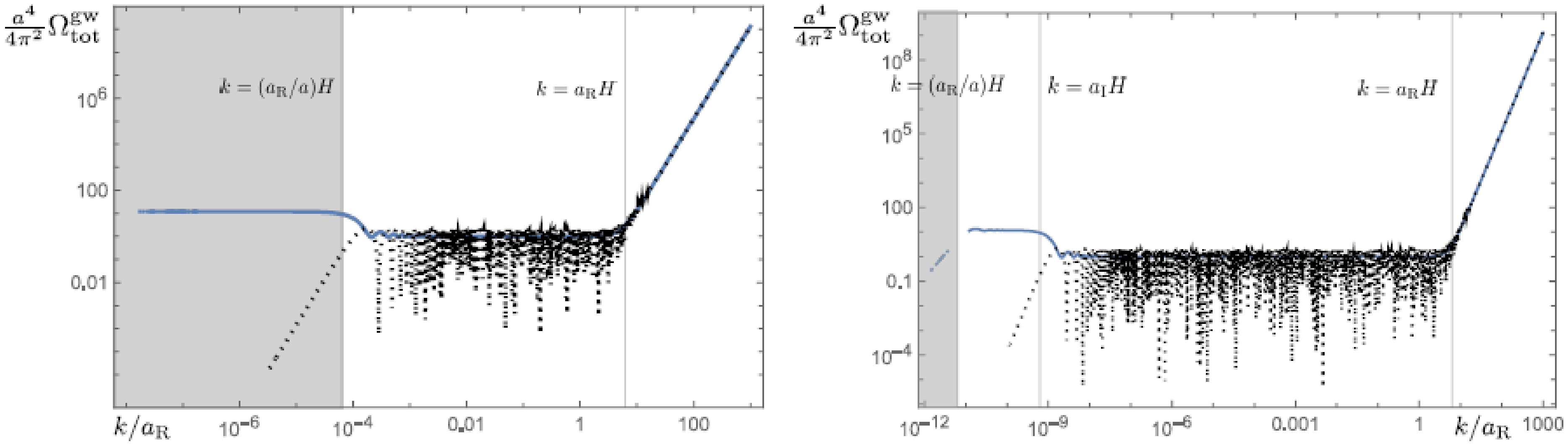



Figure 3: Power spectral density of the Isaacson stress tensor (dashed lines) evaluated at $a = 10^5 a_R$ (left) and $a = 10^{13} a_R$ (right), with $a_I = 10^{-12} a_R$ in units where $H$ is set to $2\pi$. The blue line is the comparison to the spectral density of the improved stress tensor, plotted in Fig. 4. The gray shaded regions correspond to scales outside the domain of validity of the Isaacson stress tensor, where we note that the oscillations would not appear in its time averaged form.

# Now we can finally reconsider $N_{\rm eff}$ bounds

- In essence, this is a question about how the background gets renormalized from vacuum tensor perturbations.

- Let's reconsider the academic case of a test scalar:

$$\rho_{\rm div}^{\rm log} = \frac{H^4(1+e^{-4\mathcal{N}_{\rm tot}})}{8\pi^2(a/a_{\rm R})^4}\left[\frac{1}{\delta_{\rm UV}} + 1 - \gamma_E + \log\left(\frac{\mu}{H}\right)\right]$$

$$S = S_{\rm EH} + S_{\rm bg} + S_\phi + S_{\rm ct}$$

$$\frac{1}{8\pi G_B}\left(R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}\right) = T^{\rm bg}_{\mu\nu} + T^{\phi}_{\mu\nu} + T^{\rm ct}_{\mu\nu}$$

$$\frac{1}{8\pi G_B}\frac{3H^2}{(a/a_R)^4} = \rho^{\rm cl}_{\rm bg} + \frac{H^4(1+e^{-4\mathcal{N}_{\rm tot}})}{8\pi^2(a/a_{\rm R})^4}\left[\frac{1}{\delta_{\rm UV}} + 1 - \gamma_E + \log\left(\frac{\mu}{H}\right)\right] + \rho_{\rm ct} + \rho_{\phi,\rm finite}$$

$$\rho_{\rm ct} = \frac{3H^2}{(a/a_R)^4}\left(\frac{B_{-1}}{\delta_{\rm UV}} + B_0\right) \qquad B_{-1} = -\frac{H^2(1+e^{-4\mathcal{N}_{\rm tot}})}{24\pi^2}$$

# Now we can finally reconsider $N_{\text{eff}}$ bounds

- Do we (multiplicatively) renormalize G_N, or (additively) renormalize background matter sector?

- Shifted tadpole condition: doesn't matter!

$$\frac{1}{8\pi G_N(\mu)} = \frac{1}{8\pi G_B} - B_0 - \frac{H^2}{24\pi^2}(1+e^{-4\mathcal{N}_{\text{tot}}})\left\{1-\gamma_E+\log\left(\frac{\mu}{H}\right)\right\}$$

- Fix G_N via measurement (Cavendish) at some scale:

$$\frac{1}{8\pi G_B} = \frac{1}{8\pi G_N(\mu_*)} + B_0 + \frac{H^2}{24\pi^2}(1+e^{-4\mathcal{N}_{\text{tot}}})\left\{1-\gamma_E+\log\left(\frac{\mu_*}{H}\right)\right\}$$

- Use this to eliminate G_B in the above, so that:

$$\frac{1}{8\pi G_N(\mu)} = \frac{1}{8\pi G_N(\mu_*)} - \frac{H^2}{24\pi^2}(1+e^{-4\mathcal{N}_{\text{tot}}})\log\left(\frac{\mu}{\mu_*}\right)$$

# Now we can finally reconsider $N_{\rm eff}$ bounds

- We finally obtain:

$$G_{\mu\nu} = \frac{1}{M_{\rm pl}^2} T_{\mu\nu}^{\rm bg,shift} \left[1 - \frac{H^2(1+e^{-4\mathcal{N}_{\rm tot}})}{24\pi^2 M_{\rm pl}^2} \log\left(\frac{\mu}{\mu_*}\right)\right]^{-1}$$

- Can't go any further for test scalars, but for GWs, we have a classically evolving background (FRLW), and can fix one more renormalization condition to fix shifted tadpole:

$$\frac{3H^2}{(a/a_R)^4} = \frac{1}{M_{\rm pl}^2}\left(\rho_{\rm bg}^{\rm cl} + \rho_{\rm gw,finite}\right)\left[1 - \frac{H^2(1+e^{-4\mathcal{N}_{\rm tot}})}{12\pi^2 M_{\rm pl}^2} \log\left(\frac{\mu}{\mu_*}\right)\right]^{-1}$$
$$= \frac{\rho_{\rm bg}^{\rm cl}}{M_{\rm pl}^2}(1+\delta_{\rm r})\left[1 - \frac{H^2(1+e^{-4\mathcal{N}_{\rm tot}})}{12\pi^2 M_{\rm pl}^2} \log\left(\frac{\mu}{\mu_*}\right)\right]^{-1},$$

- But this is indistinguishable from rescaling scale factor at reheating. i.e. all one finds is a shift in the temperature-redshift relation.

# Can repeat the exercise in a fully covariant formalism

$$S_{\rm gw} = S^{(2)} + S_{\rm gb} + S_{\rm gh} = \frac{\kappa^2}{2}\int d^4x\sqrt{-g}\left[\frac{1}{2}h_{\rho\sigma}\Box h^{\rho\sigma} - \frac{1}{4}h\Box h + R^{\beta}{}_{\rho\alpha\sigma}h_{\beta}{}^{\alpha}h^{\rho\sigma} + h_{\alpha}{}^{\rho}h^{\alpha\sigma}R_{\rho\sigma} - hh^{\rho\sigma}R_{\rho\sigma} - \frac{1}{2}h^{\rho\sigma}h_{\rho\sigma}R \right.$$
$$\left. + \frac{1}{4}hhR + \bar{\eta}^{\mu}\left(g_{\mu\nu}\Box - R_{\mu\nu}\right)\eta^{\nu}\right].$$

- Divergences that need to be regularized:

$$\langle S_{\rm gw}\rangle_{\rm div} = \frac{1}{4\pi^2}\lim_{\sigma^{\mu}\to 0}\int d^4x\sqrt{-g}\left[\frac{11}{6\sigma}R + \ln(\Lambda^2\sigma)\left(\frac{1}{6}R_{\mu\nu}R^{\mu\nu} - \frac{11}{24}R^2 - \frac{23}{24}\Box R\right)\right]$$

$$\langle S\rangle = S_{\rm EH} + S_{\rm RD} + S_{\rm ct} + \langle S_{\rm gw}\rangle$$

$$S_{\rm ct} = \frac{1}{4\pi^2}\lim_{\sigma^{\mu}\to 0}\int d^4x\sqrt{-g}\Big[\alpha_1(\sigma,\mu)R + \alpha_2(\sigma,\mu)R_{\mu\nu}R^{\mu\nu} + \alpha_3(\sigma,\mu)R^2 + \alpha_4(\sigma,\mu)\Box R\Big]$$

# Can repeat the exercise in a fully covariant formalism

$$\langle S\rangle = S_{\rm EH} + S_{\rm RD} + S_{\rm ct} + \langle S_{\rm gw}\rangle \qquad S_{\rm RD} = \int d^4x\sqrt{-g}P(X) \qquad P(X) = X^2, \text{ with } X := -\tfrac{1}{2}g^{\mu\nu}\partial_\mu\psi\partial_\nu\psi$$

$$\langle S\rangle = \int d^4x\sqrt{-g}\left[\frac{1}{16\pi G(\mu)}R + \bar{\alpha}_2(\mu)R_{\mu\nu}R^{\mu\nu} + \bar{\alpha}_3(\mu)R^2 + \bar{\alpha}_4(\mu)\Box R + P(X_B)\right] + \langle S_{\rm gw}\rangle_{\rm fin}$$

$$\frac{1}{16\pi G(\mu)} = \frac{1}{16\pi G_B} + \frac{\alpha_1^{\rm F}(\mu)}{4\pi^2},$$

$$\bar{\alpha}_2(\mu) = \frac{1}{4\pi^2}\left[\frac{1}{6}\log\frac{\Lambda^2}{\mu^2} + \alpha_2^{\rm F}(\mu)\right],$$

$$\bar{\alpha}_3(\mu) = \frac{1}{4\pi^2}\left[-\frac{11}{24}\log\frac{\Lambda^2}{\mu^2} + \alpha_3^{\rm F}(\mu)\right],$$

$$\bar{\alpha}_4(\mu) = \frac{1}{4\pi^2}\left[-\frac{23}{24}\log\frac{\Lambda^2}{\mu^2} + \alpha_4^{\rm F}(\mu)\right].$$

# Can repeat the exercise in a fully covariant formalism

$$\langle S \rangle = S_{\rm EH} + S_{\rm RD} + S_{\rm ct} + \langle S_{\rm gw} \rangle \qquad S_{\rm RD} = \int d^4x \sqrt{-g} P^{\rm bg}(X) \qquad T^{{\rm bg}\,\mu}{}_{\nu}(X_B) = \delta^{\mu}{}_{\nu} P^{\rm bg} - P^{\rm bg}{}_{,X} \partial^{\mu}\psi_B \partial_{\nu}\psi_B$$

$$\langle S \rangle = \int d^4x \sqrt{-g} \left[ \frac{1}{2} M^2(\mu) R + \widetilde{P}^{\rm bg}(X_B, \mu) \right] + \langle S_{\rm gw} \rangle_{\rm fin} \qquad \widetilde{P}^{\rm bg}(X_B, \mu) := X_B^2 + 12 X_B^4 \frac{\bar{\alpha}_2(\mu)}{M^4(\mu)};$$

$$\widetilde{T}^{{\rm bg}\,\mu}{}_{\mu} = -48 X_B^4 \frac{\bar{\alpha}_2(\mu)}{M^4(\mu)}$$

Einstein gravity is not scale invariant – makes sense that the effective stress tensor has a trace. Foliation specific computation before suggests canceled by adiabatic vacuum contributions.

Repeat as before, contributions that redshift like radiation have the same interpretation…

# Can repeat the exercise in a fully covariant formalism

$$\frac{1}{8\pi G(\mu)}\left(R_{\mu\nu}-\frac{1}{2}Rg_{\mu\nu}\right)=\widetilde{T}^{\rm bg}_{\mu\nu}+\langle T^{\rm gw,fin}_{\mu\nu}\rangle$$

$$8\pi G_N(\mu)=\frac{1}{M_{\rm pl}^2}\left[1+\frac{\alpha_1^{\rm F}(\mu)-\alpha_1^{\rm F}(\mu_*)}{2\pi^2M_{\rm pl}^2}\right]^{-1}$$

$$3H_{\rm R}^2(3\omega_{\rm R}-1)=-48X_B^4\frac{\bar{\alpha}_2(\mu_{\rm R})}{M_{\rm pl}^6}+\frac{\langle T^{\rm gw,fin}\rangle}{M_{\rm pl}^2}$$

$$3H^2=\frac{1}{M_{\rm pl}^2}\left(\widetilde{\rho}_{\rm bg}+\rho^{\rm rd}_{\rm gw,fin}\right)\left[1+\frac{\alpha_1^{\rm F}(\mu)-\alpha_1^{\rm F}(\mu_*)}{2\pi^2M_{\rm pl}^2}\right]^{-1}\approx\frac{\widetilde{\rho}_{\rm bg}}{M_{\rm pl}^2}\left(1+\delta_{\rm Z}\right)$$

$$\psi:=(1+\delta_{\rm Z})^{1/4}\,\psi_B$$

Newton’s constant fixed via Cavendish type experiment at laboratory scales, does not run at cosmological scales. Fix equation of state parameter with another measurement.

Repeat as before, effects wave function renormalization…

# So what?

- $N_{\rm eff}$ bounds are physically about entropy to baryon ratio of light species that have frozen out. It makes physical sense that vacuum fluctuations do not contribute to this.

- To anyone interested in measuring GHz, THz GW backgrounds: $N_{\rm eff}$ bounds do not kill your science case.

- Keep calm, regularize, *and* renormalize.

# So what?

- $N_{\rm eff}$ bounds are physically about entropy to baryon ratio of light species that have frozen out. It makes physical sense that vacuum fluctuations do not contribute to this.
- To anyone interested in measuring GHz, THz GW backgrounds: $N_{\rm eff}$ bounds do not kill your science case.
- Keep calm, regularize, *and* renormalize.

# Thank you for listening!