Tunneling rate at finite temperature

MASAHIDE YAMAGUCHI

(Institute for Basic Science)

1/April/2026@University of Helsinki

Yutaro Shoji, MY, arXiv:2511.05950, to appear in Adv. Theor. Math. Phys

$$c = \hbar = 1, \quad M_G = 1/\sqrt{8\pi G} \sim 2.4 \times 10^{18} \mathrm{GeV}.$$

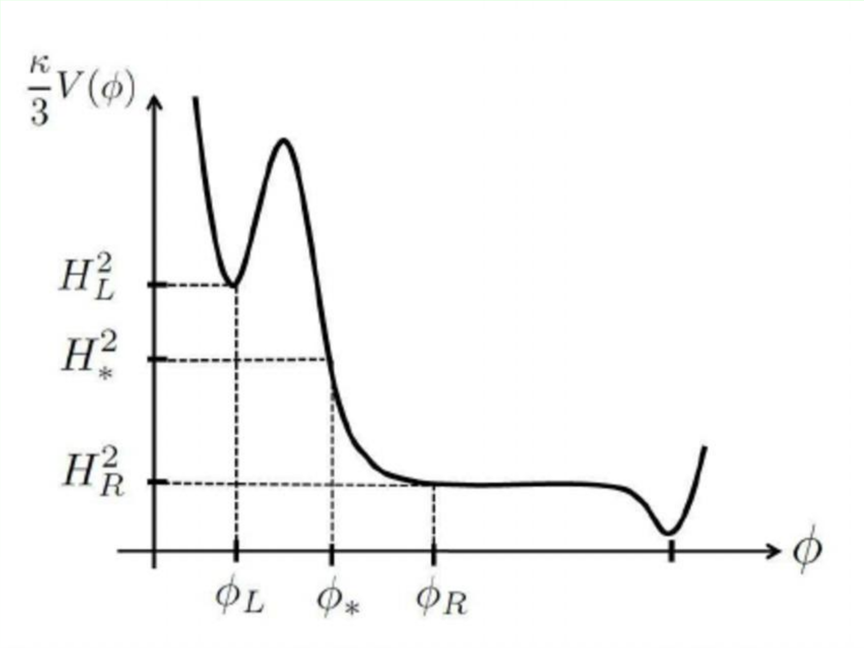
Contents

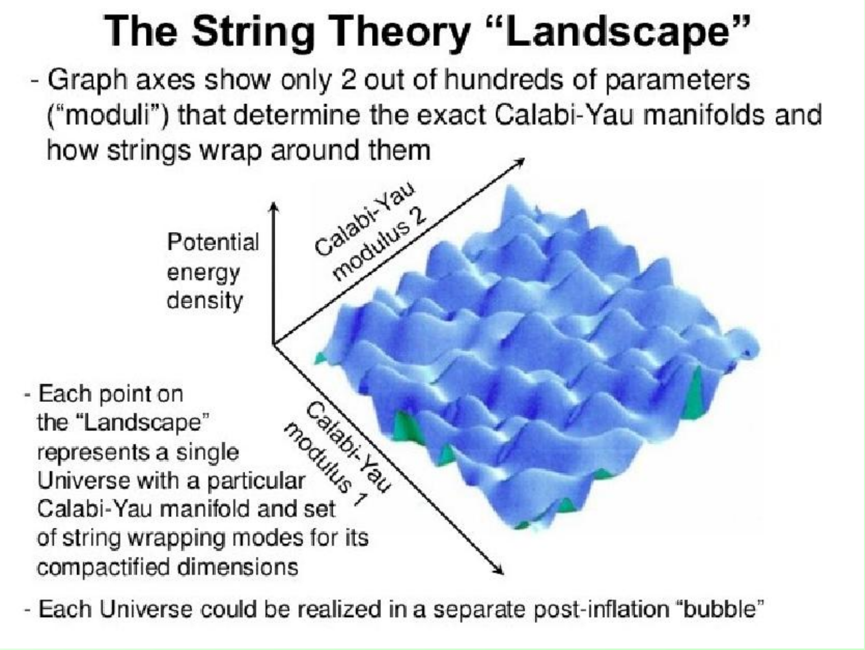
Introduction

False vacuum decay in cosmology

Renewed interest :

(Open) Inflation model,
string landscape,
Baryogenesis from (electroweak) first order phase transition
GW from first order phase transition, …



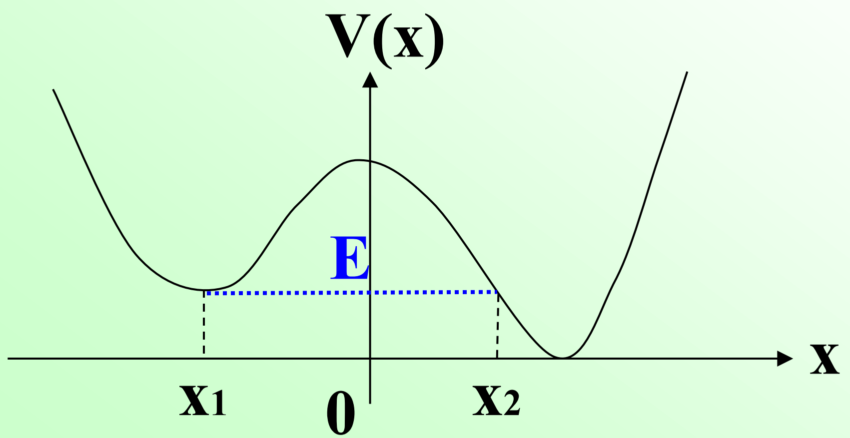


Credit: Takahiro Tanaka **Credit: Michele Nardelli**

How to estimate the tunneling rate ?

Quantum tunneling at zero temperature

1-dim point particle system with the following potential :



EOM : $\frac{d^2x}{dt^2} = -V'(x) \longrightarrow \frac{1}{2}\left(\frac{dx}{dt}\right)^2 + V(x) = E.$ **(m = 1 for simplicity)**

If E < V(0), there is no classical dynamics for the transition from x_1 to x_2.

Especially, a particle sitting on x_1 stays there forever classically.

But, quantum mechanically, it can decay (penetrate) into x_2.

WKB approximation

(m = 1 for simplicity)

Schrödinger equation : $$\frac{\hbar^2}{2}\psi''(x) + (E - V(x))\psi(x) = 0 \qquad \left(' = \frac{d}{dx}\right)$$

$$\psi(x) \equiv \exp\left(i\frac{S(x)}{\hbar}\right) \quad \Rightarrow \quad S'(x)^2 - i\hbar S''(x) - 2(E - V(x)) = 0$$

WKB approximation : $$S(x) = S_0(x) + \frac{\hbar}{i}S_1(x) + \left(\frac{\hbar}{i}\right)^2 S_2 + \cdots$$

$$\Rightarrow \begin{cases} \mathcal{O}(\hbar^0) : \; S_0'(x)^2 - 2(E - V(x)) = 0 \\ \mathcal{O}(\hbar^1) : \; 2S_0'(x)S_1'(x) + S_0''(x) = 0 \end{cases}$$

Hamilton-Jacobi eq.

For E – V(x) < 0, S_0 must be imaginary (no classical dynamics).

$$\begin{cases} S_0 = \pm i\int\sqrt{2(V(x) - E)}dx \\ S_1 = -\frac{1}{4}\log\left[2i(V(x) - E)\right] \end{cases} \quad \Rightarrow \quad \psi(x) = \frac{\text{const}}{[2(V(x) - E)]^{1/4}}\exp\left(\pm\frac{1}{\hbar}\int\sqrt{2(V(x) - E)}dx\right)$$

$\Rightarrow$ **Transition rate :** $$T \sim \exp\left(-\frac{2}{\hbar}\int_{x_1}^{x_2} dx\sqrt{2\left(V(x) - E\right)}\right)$$

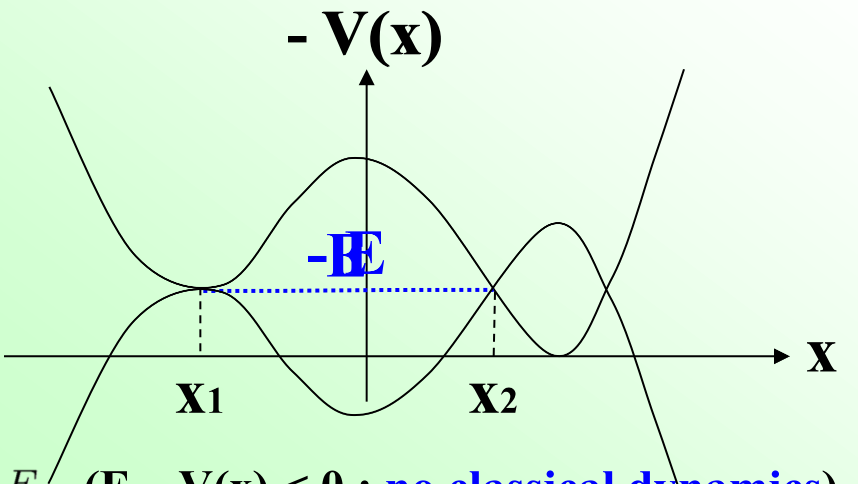
Matching at turning points x1 and x2 with V(x)=E

Tunneling process in quantum mechanics

Transition rate (through WKB method)：

$$T \propto \exp\left(-2\int_{x_1}^{x_2} dx\sqrt{2\left(V(x)-E\right)}\right)$$

$(\hbar = 1)$



Another way:

EOM : $\frac{d^2x}{dt^2} = -V'(x) \longrightarrow \frac{1}{2}\left(\frac{dx}{dt}\right)^2 + V(x) = E.$ **(E – V(x) < 0 : no classical dynamics)**

(Wick rotation: τ= it)

$\frac{d^2x}{d\tau^2} = +V'(x) \longrightarrow \frac{1}{2}\left(\frac{dx}{d\tau}\right)^2 - V(x) = -E.$ **(E – V(x) < 0 : classical dynamics)**

$$2\int_{x_1}^{x_2} dx\sqrt{2\left(V(x)-E\right)} = 2\int_{x_1}^{x_2} dx\frac{dx}{d\tau} = 2\int_{-\infty}^{0} d\tau\left(\frac{dx}{d\tau}\right)^2 = \int_{-\infty}^{\infty} d\tau\left[\frac{1}{2}\left(\frac{dx}{d\tau}\right)^2 + V(x) - E\right]$$

$$= S_E[x_B] - S_E[x_1]. \qquad \left(S_E[x] = \int_{-\infty}^{\infty} d\tau\left[\frac{1}{2}\left(\frac{dx}{d\tau}\right)^2 + V(x)\right]\right)$$

A "bouncing" solution starting from x1, reaching x2, and going back to x1.

By finding this bouncing solution and evaluating its Euclidean action, we can estimate the rate for tunneling through a barrier.

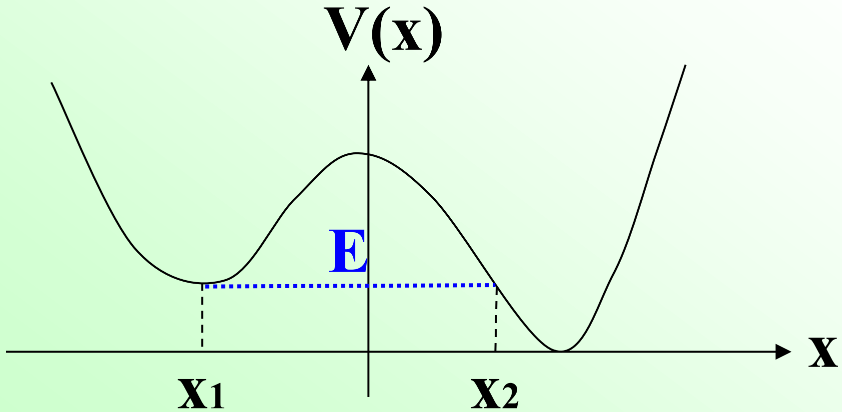
Tunneling process in quantum field theory

(Coleman 1977, Callan & Coleman 1977)

Transition rate (through WKB method) :

$$T \propto \exp\left(-2\int_{x_1}^{x_2} dx\sqrt{2\left(V(x)-E\right)}\right)$$

$$= \exp\left[-\left(S_E[x_B(\tau)] - S_E[x_1]\right)\right]$$



Straightforward generalization to many-body system, $\{q_i\}$

$$S_E[q_i(\tau)] = \int d\tau\left[\sum_i \frac{1}{2}\left(\frac{dq_i}{d\tau}\right)^2 + V(q_i)\right]$$

A path with the least (Euclidean) action is dominant because of $T \propto e^{-S_E}$.

Further generalization to a field theory, φ(x)

$$S_E[\phi(x)] = \int d^4x_E\left[\frac{1}{2}\delta_E^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + V(\phi)\right] = \int d\tau\int d^3x\left[\frac{1}{2}\left(\frac{\partial\phi}{\partial\tau}\right)^2 + \frac{1}{2}(\partial_i\phi)^2 + V(\phi)\right]$$

(Effective potential includes spatial derivatives)

N.B. There is a crucial difference (gradient term) between QM and QFT.

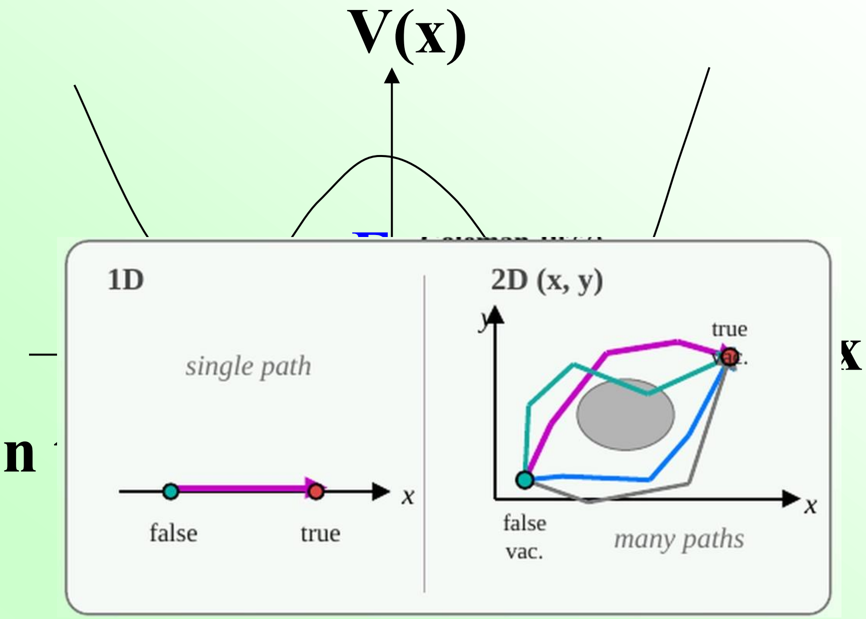
Tunneling process in quantum field theory

(Coleman 1977, Callan & Coleman 1977)

Transition rate (through WKB method) :

$$T \propto \exp\left(-2\int_{x_1}^{x_2} dx \sqrt{2\left(V(x)-E\right)}\right)$$

$$= \exp\left[-\left(S_E[x_B(\tau)] - S_E[x_1]\right)\right]$$



Straightforward generalization

$$S_E[q_i(\tau)] = \int d\tau \left[\sum_i \frac{1}{2}\left(\frac{dq_i}{d\tau}\right)^2 + V(q_i)\right]$$

A path with the least (Euclidean) action is dominant because of $T \propto e^{-S_E}$.

Further generalization to a field theory, φ(x)

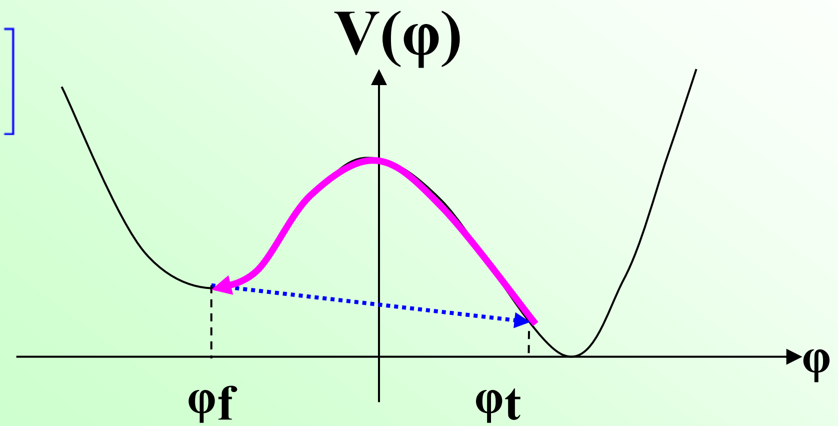
$$S_E[\phi(x)] = \int d^4x_E \left[\frac{1}{2}\delta_E^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + V(\phi)\right] = \int d\tau \int d^3x \left[\frac{1}{2}\left(\frac{\partial\phi}{\partial\tau}\right)^2 + \frac{1}{2}(\partial_i\phi)^2 + V(\phi)\right]$$

(Effective potential includes spatial derivatives)

N.B. There is a crucial difference (gradient term) between QM and QFT.

Tunneling process in quantum field theory II

Decay rate : $\Gamma \propto \exp\left[-\left(S_E[\phi_B(x)] - S_E[\phi_f(x)]\right)\right]$



$$S_E[\phi(x)] = \int d^4x_E \left[\frac{1}{2}\delta_E^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + V(\phi)\right]$$
$$= \int d\tau \int d^3x \left[\frac{1}{2}\left(\frac{\partial\phi}{\partial\tau}\right)^2 + \frac{1}{2}(\partial_i\phi)^2 + V(\phi)\right]$$

EOM ⇒ $-\delta_E^{\mu\nu}\partial_\mu\partial_\nu\phi + V'(\phi) = 0.$

Coleman et al. proved that the solution with O(4) symmetry gives the least action.

⇒ $\ddot{\phi} + \frac{3}{\rho}\dot{\phi} = V'(\phi), \quad \rho = \sqrt{\tau^2 + \boldsymbol{x}^2}. \qquad \left(\dot{} = \frac{d}{d\rho}\right)$

with the boundary conditions, $\frac{d\phi}{d\rho}(0) = 0, \quad \lim_{\rho\to\infty}\phi(\rho) = \phi_f.$

Decay rate : $\Gamma = A\exp(-B).$

one-loop, that is, Gaussian term

$$A = \left(\frac{B}{2\pi}\right)^2 \left|\frac{\det'\left(S_E''[\phi_B]\right)}{\det\left(S_E''[\phi_f]\right)}\right|^{-\frac{1}{2}}, \qquad B = S_E[\phi_B] - S_E[\phi_f].$$

(Coleman 1977, Callan & Coleman 1977)

CGM theorem

(Coleman, Glaser, and Martin, Commun. math. Phys. 58, 211-221 (1978))

- **In D(>2)-dimensional Euclidean space,**

  EOM: $\Delta\phi - V'(\phi) = 0,$

  ➔ at least one monotone spherical O(D) solution vanishing (false vacuum) at infinity,
  (other than the trivial solution of φ= 0 (false vacuum))
  if the potential V satisfies some conditions.

- **This solution has the least Euclidean action,**

$$S = \int d^D x \left[\frac{1}{2}(\partial\phi)^2 + V(\phi)\right],$$

among any other solution vanishing at infinity.
If the other solution is not both spherical O(D) and monotone, this action is strictly less than that of the other solution.

Conditions for the potential

i) V is continuously differentiable for all φ.
ii) V(0) = V'(0) = 0. (φ = 0 corresponds to a false vacuum.)
iii) V is somewhere negative.
iv) there exist positive numbers a, b, α, and β such that

$$\alpha < \beta < 2D/(D-2),$$

$$\text{with} \quad V - a|\phi|^{\alpha} + b|\phi|^{\beta} \geq 0.$$

Coleman, Glaser, Martin proved that the solution with O(4) symmetry gives the least Euclidean action when gravity is not taken into account.

Coleman, Glaser, and Martin, Commun. math. Phys. 58, 211-221 (1978)

None proved that the solution with maximal symmetry gives the least Euclidean action (≒ the most probable process) when gravity is taken into account.

↑

Oshita, Shoji, MY justified this claim through AdS/CFT correspondence for specific (large & thin wall) cases.

(e-Print: 2308.02159 [hep-th], PTEP 2024 (2024) 6, 063E01)

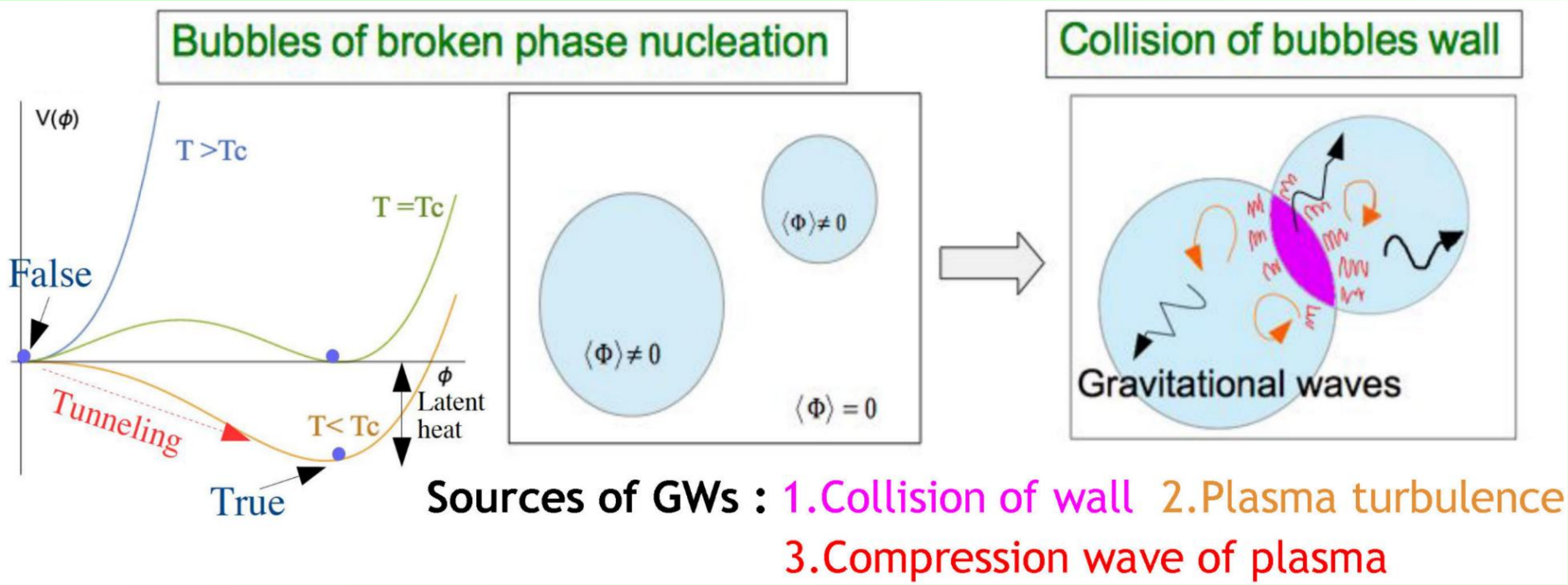
This is still an open question for a general case!!

(But, today's topic is different.)

How to estimate the tunneling rate
“at finite temperature” ?

Why is this topic so important ?

Gravitational waves from first order phase transitions



https://conference-indico.kek.jp/event/51/contributions/1118/attachments/720/754/KEKPH_2018_winter_Katsuya.pdf

Tunneling rate is quite important for making the prediction of GWs.

Tunneling rate and the prediction of gravitational waves from first order phase transitions

★ $$B \equiv \frac{d}{dt} \ln \Gamma(t) \Big|_{t=t_*}$$

Γ(t) : tunneling rate /time/volume
t_*, T_* : time, temperature when Γ / H^4 ∼ 1

➡ **Time scale of phase transition : Δt ∼ 1 / B**

➡ $$f_{\rm peak} \propto \frac{B}{H_*} \frac{T_*}{M_{\rm pl}}$$

★ $$A \equiv \frac{\rho_{\rm false}(T_*) - \rho_{\rm true}(T_*)}{\rho_{\rm rad}(T_*)}$$

(A ≳ 1 : large latent heat)

➡ $$\Omega_{\rm GW} \propto \left(\frac{H_*}{B}\right)^p \frac{A^q}{(1+A)^r}.$$

(p, q, r depend on the sources)

Gravitational-wave energy density ~
"(energy density of the source) × (time during which the source is active) × (geometric factor)"

$\sim \frac{\Delta\rho}{\rho_{\rm tot}} \sim \frac{A}{1+A}$ $\sim \frac{H_*}{B}$

~ A for small A
~ 1 for large A

How to estimate the tunneling rate "at finite temperature" ?

Finite temperature QFT

Partition function : $Z(\beta) = \mathrm{Tr}\, e^{-\beta H}$ with $\beta = \frac{1}{T}$

➡ **Euclidean path integral with τ compactified (period β)**

$$Z(\beta) = \int_{\phi(\tau+\beta)=\phi(\tau)} \mathcal{D}\phi \; e^{-S_E[\phi]}$$

$$S_E = \int_0^\beta d\tau \int d^{D-1}x \left[\frac{1}{2}(\partial_\tau \phi)^2 + \frac{1}{2}|\nabla_x \phi|^2 + V(\phi)\right]$$

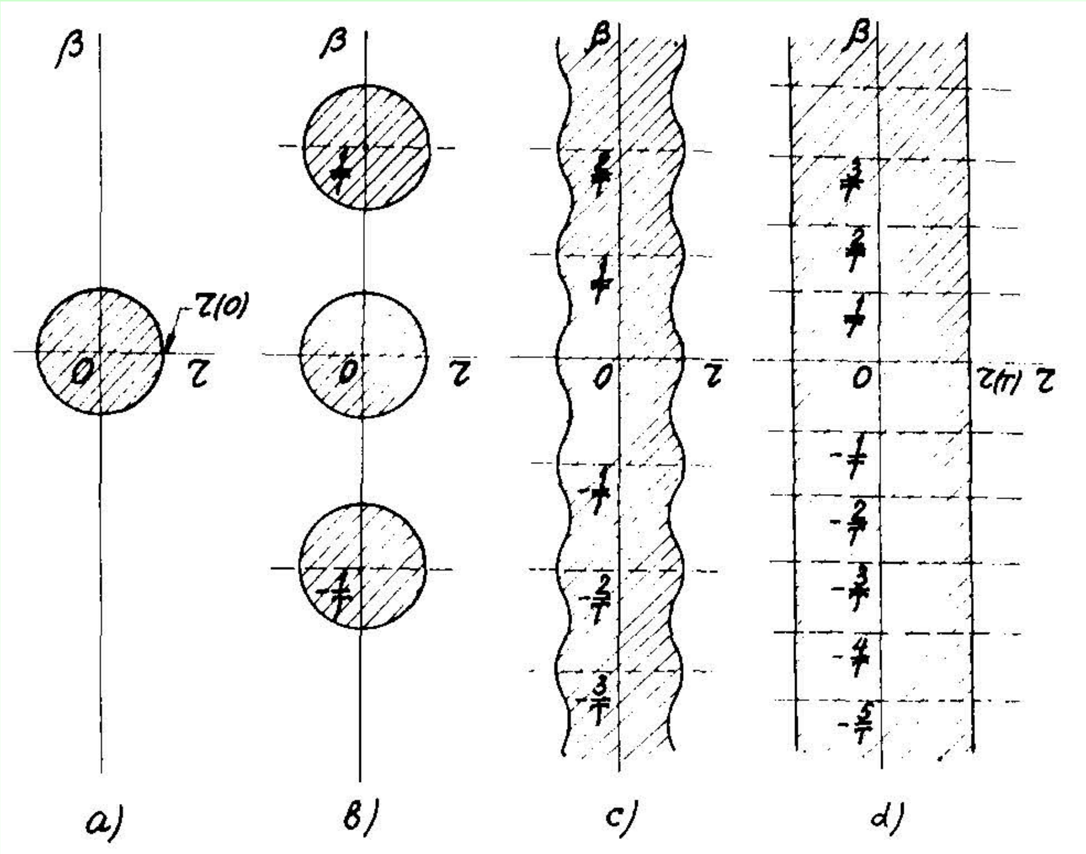
- Bosons: periodic $\phi(\tau+\beta) = \phi(\tau)$
- Fermions: anti-periodic $\psi(\tau+\beta) = -\psi(\tau)$

Finite temperature QFT lives on $R^{D-1} \times S^1$.

➔ **Saddle point solutions must be periodic.**

Tunneling rate at finite temperature

(Linde 1977, 1981)



(only spatial directions)

e)

(a) T = 0; (b) T << r(0)$^{-1}$; (c) T ~ r(0)$^{-1}$; (d) T >> r(0)$^{-1}$; (e) T ➔ ∞(β ➔0)

(r(0) : O(4)-symmetric bubble size)

O(4)(CGM) **O(3) ??? (assumption)** **O(3)(CGM)**

This talk

$$S_E \to \beta S_3$$

Does the O(3) symmetric bounce solution give the least Euclidean action even at "arbitrary" finite temperature" ?

(even if gravity effects are neglected.)

Review of proof of CGM theorem

(Coleman, Glaser, and Martin, Commun. math. Phys. 58, 211-221 (1978))

CGM theorem

(Coleman, Glaser, and Martin, Commun. math. Phys. 58, 211-221 (1978))

- **In D(>2)-dimensional Euclidean space,**

  EOM: $$\Delta\phi - V'(\phi) = 0, \qquad (☆)$$

  ➔ at least one monotone spherical O(D) solution vanishing (false vacuum) at infinity,
  (other than the trivial solution of φ= 0 (false vacuum))
  if the potential V satisfies some conditions.

- **This solution has the least Euclidean action,**

$$S = \int d^D x \left[\frac{1}{2}(\partial\phi)^2 + V(\phi)\right],$$

among any other solution vanishing at infinity.
If the other solution is not both spherical O(D) and monotone, this action is strictly less than that of the other solution.

Reduced problem

NB: 1) S is unbounded (staying at true vacuum => S ➔ - ∞)

2) Find a saddle point solution (satisfying EOM and $\varphi(\infty) = 0$) which gives the least Euclidean action

➡ Introduce "Reduced problem"

(A) Split $S = \int d^D x \left[\frac{1}{2}(\partial\phi)^2 + V(\phi)\right] \equiv T + U$

(B) Under scaling $\varphi_a(x)=\varphi(x/a)$: $T \to a^{D-2}\, T$, $U \to a^D\, U$

(C) Stationarity ⇒ $(D-2)T + D\,U = 0$ ⇒ $S = 2\,T / D$

(D) Reduced problem : Find a function satisfying $\varphi(\infty) = 0$ which minimizes T for fixed (negative) U

⇔ minimizes $R = -T^{(D/D-2)} / U$ (scale invariant)

Theorem A

Reduced problem :

Find a function satisfying φ(∞) = 0
which minimizes T for fixed (negative) U
⇔ minimizes R = -T$^{(D/D-2)}$ / U (scale invariant)

Theorem A: If a solution of the reduced problem exists, then, for chosen U, it is a solution of EOM (☆) that has action less than or equal to that of any non-trivial solution of EOM (☆).

Proof : φ' : a non-trivial solution of EOM (☆) ➔ (D-2) T[φ'] + U[φ'] = 0

$$S' = T + \lambda^2 \left(U - U[\phi']\right)$$

λ^2 : Lagrange multiplier

$$\frac{\delta S'}{\delta \phi(x)} = -\Delta\phi(x) + \lambda^2 V'(\phi(x)) = 0$$

➔ (D-2) T[φ] + λ^{2}U[φ] = 0

$$\Delta\phi_\lambda(x) + V'(\phi_\lambda(x)) = 0$$

satisfying EOM (☆)

$$\phi_\lambda(x) \equiv \phi(x/\lambda)$$

$$T[\phi_\lambda] = \lambda^{D-2} T[\phi] \le T[\phi] \le T[\phi']$$

(T[φ] ≤ T[φ'] ➔ λ^2 ≤ 1)

$$S[\phi_\lambda] = \frac{2}{D} T[\phi_\lambda] \le \frac{2}{D} T[\phi'] = S[\phi']$$

Theorem B

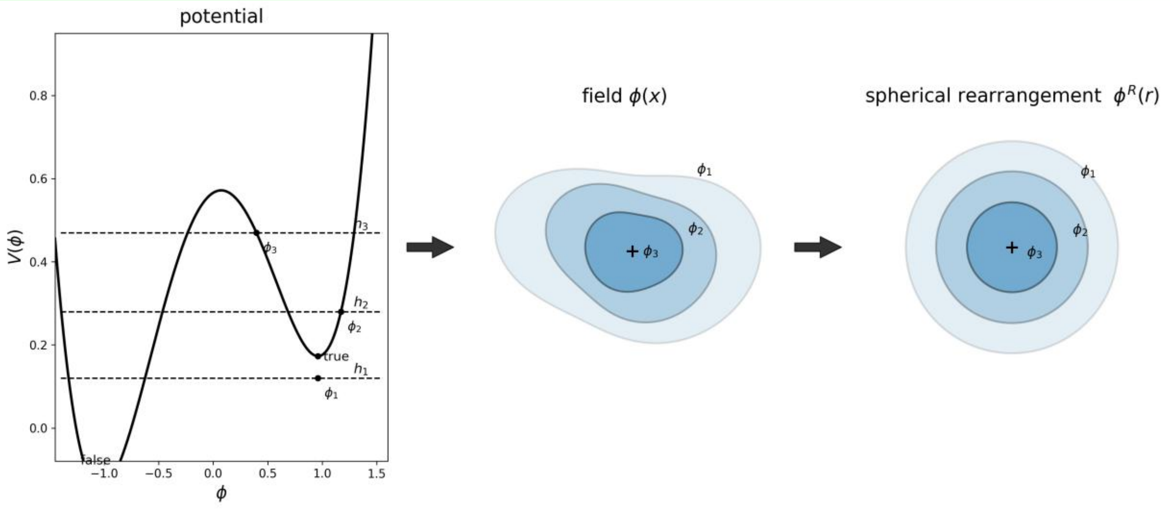
Theorem B : There exists at least one solution to the reduced problem. All solutions to the reduced problem are spherically O(4) symmetric and monotone.

- **Build a minimizing sequence**
  - Choose $\{\varphi_n\}$ with fixed $U[\varphi_n] = U_0 < 0$ and $T[\varphi_n] \to \inf T$
  - Reduce to nonnegative (OR nonpositive) functions
- **Spherical Rearrangement → monotone & spherical O(4)**
  - U: invariant, T: nonincreasing ➔ can set spherical & decreasing
  - Equality case ⇔ only if original function is already spherical & monotone
- **Compactness & limit**
  - Conditions for potential ➔ Uniform bounds on several integrals ➔ Ascoli (local uniform conv.), Fatou + weak lower semicontinuity
  - Limit $\varphi = \{\varphi_n\}(n \Rightarrow \infty)$ attains the minimum of R ➔ solves reduced problem; hence spherical & monotone

Theorem A + Theorem B ➔ CGM Theorem

Theorem B

Theorem B : There exi
problem. All solutions t
O(4) symmetric and mo



- **Build a minimizing s**
  - Choose {φ_n} with fixe
  - Reduce to nonnegativ

- **Spherical Rearrangement → monotone & spherical O(4)**
  - U: invariant, T: nonincreasing ➔ can set spherical & decreasing
  - Equality case ⇔ only if original function is already spherical & monotone

- **Compactness & limit**
  - Conditions for potential ➔ Uniform bounds on several integrals
    ➔ Ascoli (local uniform conv.), Fatou + weak lower semicontinuity
  - Limit φ = {φ_n}($n \Rightarrow \infty$) attains the minimum of R
    ➔ solves reduced problem; hence spherical & monotone

Theorem A + Theorem B ➔ CGM Theorem

Conditions for the potential

i) V is continuously differentiable for all φ.
ii) V(0) = V'(0) = 0. (φ = 0 corresponds to a false vacuum.)
iii) V is somewhere negative.
iv) there exist positive numbers a, b, α, and β such that

$$\alpha < \beta < 2D/(D-2),$$

$$\text{with} \quad V - a|\phi|^{\alpha} + b|\phi|^{\beta} \geq 0.$$

(Arbitrary) finite temperature case

Our claim (theorem)

- **In R^{D-1} x S^1 (D > 3) space,**

  EOM: $$\partial_\tau^2\phi + \Delta_x\phi - V'(\phi) = 0,$$

  ➔ at least one monotone spherical O(D-1) solution in the spatial directions with vanishing (false vacuum) at spatial infinity,
  (other than the trivial solution of φ= 0 (false vacuum))

  if the potential V satisfies some conditions.

- **This solution has the least Euclidean action,**

$$S_E = \int_0^\beta d\tau \int d^{D-1}x \left[\frac{1}{2}(\partial_\tau\phi)^2 + \frac{1}{2}|\nabla_x\phi|^2 + V(\phi)\right]$$

  among any other solution vanishing at infinity.
  If the other solution is not both spherical O(D-1) and monotone, this action is strictly less than that of the other solution.

Conditions for the potential

i) V is Borel measurable and lower semicontinuous (and C^1).
ii) V(0) = 0. (φ = 0 corresponds to a false vacuum.)
iii) V is somewhere negative.
iv) there exist positive numbers a, b, α, and β such that

$$\alpha < \beta < 2D/(D-2),$$

$$\text{with} \quad V - a|\phi|^{\alpha} + b|\phi|^{\beta} \geq 0.$$

Main Difference

Zero temperature

$$S = \int d^D x \left[\frac{1}{2}(\partial\phi)^2 + V(\phi)\right] \equiv T + U$$

(D−2)T + D U = 0 ⇒ S = 2 T / D

Reduced problem :
Find a function satisfying φ(∞) = 0
which minimizes T for fixed (negative) U
⇔ minimizes R = -T$^{(D/D-2)}$ / U (scale invariant)

Spherical Rearrangement
→ monotone & spherical O(D=4)

U: invariant, T: nonincreasing

➔ can set spherical O(D=4) & decreasing

Equality case ⇔ only if original function is already spherical & monotone

Finite temperature

$$S = \int_0^\beta d\tau \int d^{D-1}x \left[\frac{1}{2}(\partial_\tau\phi)^2 + \frac{1}{2}|\nabla_x\phi|^2 + V(\phi)\right]$$
$$\equiv K + T + U$$

(D−3)T + (D-1) (U+K) = 0 ⇒ S = 2T/(D-1)

Reduced problem :
Find a function satisfying φ(τ, r=∞) = 0
which minimizes T for fixed (negative) U and for K < -U
⇔ minimizes R = -T$^{(D-1/D-3)}$ / (U+K)

Steiner Rearrangement (symmetrization)
→ monotone & spherical O(D-1=3)
(Brock & Solynin 2000, Cianchi & Fusco 2006, Capriani 2011)

U: invariant, T & K: nonincreasing

➔ can set spherical O(D-1=3) & decreasing

Equality case ⇔ only if original function is already spherical & monotone

Main Difference

Zero temperature

$$S = \int d^D x \left[\frac{1}{2}(\partial\phi)^2 + V(\phi)\right] \equiv T + U$$

$(D-2)T + D\,U = 0 \;\Rightarrow\; S = 2\,T / D$

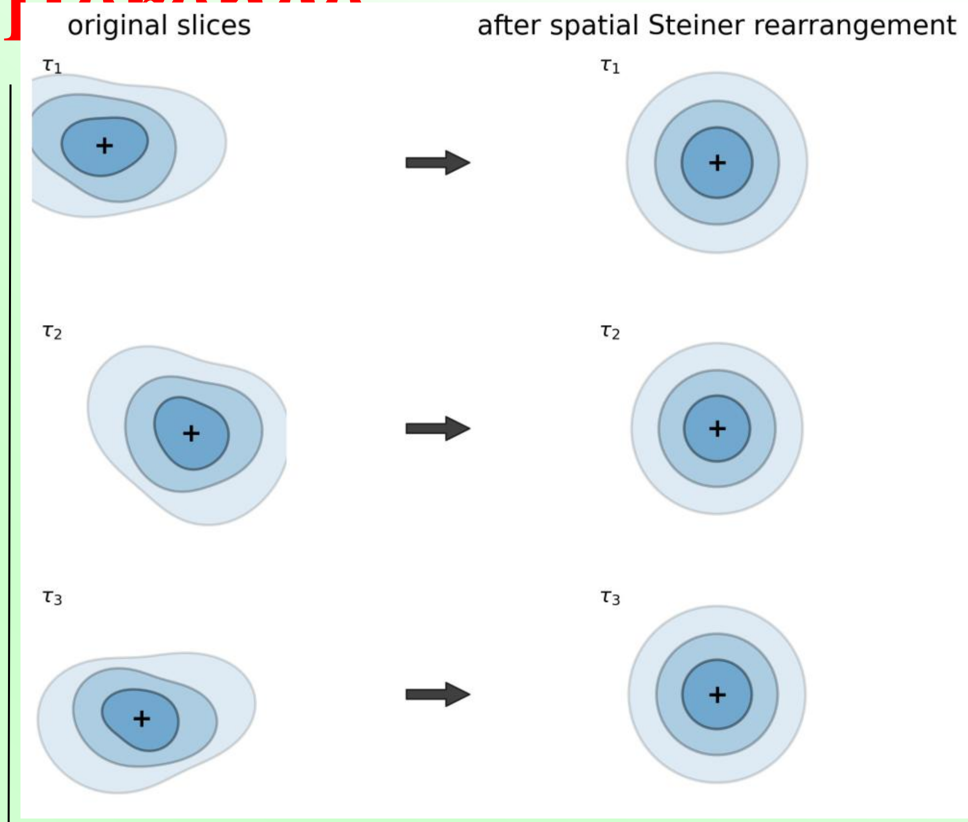
Reduced problem :
Find a function satisfying $\varphi(\infty) = 0$
which minimizes T for fixed (negative) U
⇔ minimizes $R = -T^{(D/D-2)} / U$ (scale invariant)

Spherical Rearrangement
→ monotone & spherical O(D=4)

U: invariant, T: nonincreasing

➔ can set spherical O(D=4) & decreasing

Equality case ⇔ only if original function is already spherical & monotone



Steiner Rearrangement (symmetrization)
→ monotone & spherical O(D-1=3)
(Brock & Solynin 2000, Cianchi & Fusco 2006, Capriani 2011)

U: invariant, T & K: nonincreasing

➔ can set spherical O(D-1=3) & decreasing

Equality case ⇔ only if original function is already spherical & monotone

Summary

By extending the CGM theorem for zero temperature case, we rigorously prove that

for a broad class of scalar potentials,
at arbitrary finite temperature,

any saddle-point configuration with the least action is necessarily spherical O(D−1=3) symmetric and monotonic in the spatial directions.