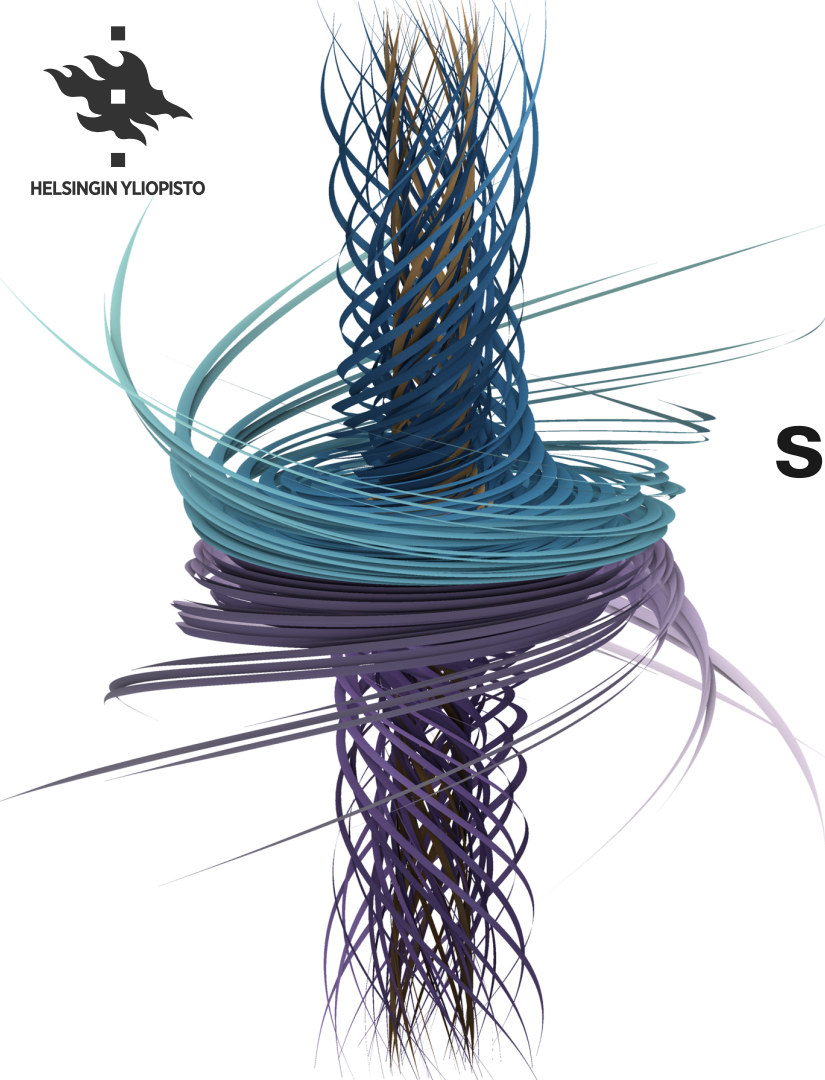




OpenAI's (possible) solution to the Navier– Stokes Millennium Problem

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Navier-Stokes equations

Velocity $\mathbf{u}(\mathbf{x}, t)$ (and pressure p) on $\mathbb{R}^3 \times [0, \infty)$ we have:

Momentum equation

$$\frac{\partial \mathbf{u}}{\partial t} + \overset{\text{Non-linear}}{\mathbf{u} \cdot \nabla \mathbf{u}} - \overset{\text{Diffusion}}{\frac{1}{\text{Re}} \nabla^2 \mathbf{u}} + \nabla p = \overset{\text{Optional forcing}}{\mathbf{f}_{\text{ext}}}$$

Pressure; Lagrange multiplier;
Instantaneous, non-local elliptic correction
 $\nabla^2 p = -\partial_i u_j \partial_j u_i + \nabla \cdot \mathbf{f}_{\text{ext}}$

Incompressibility

$$\nabla \cdot \mathbf{u} = 0$$

Describes dynamics of incompressible, viscous fluid

Degrees of freedom

4 unknown fields: u_x, u_y, u_z, p

4 independent variables: x, y, z, t

4 equations

$$\begin{aligned} 3 \times \frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} - \frac{1}{\text{Re}} \nabla^2 \mathbf{u} + \nabla p &= \mathbf{f}_{\text{ext}} \\ 1 \times \nabla \cdot \mathbf{u} &= 0 \end{aligned}$$

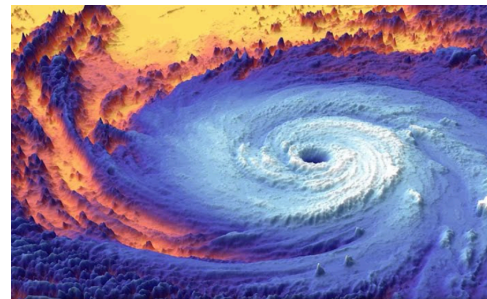
2 dynamical degrees of freedom: 3 velocity components - 1 divergence constraint

2 free initial data: $\mathbf{u}_0(\mathbf{x}) = \nabla \times [\Psi(\mathbf{x})\hat{\mathbf{e}}_3] + \nabla \times \nabla \times [\Phi(\mathbf{x})\hat{\mathbf{e}}_3]$ (Mie decomposition)

Toroidal
swirl

Poloidal
flow

Vortices in 2D flows

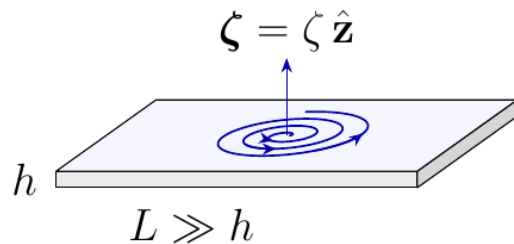


2D Navier-Stokes

$$\mathbf{v} = (v_x(x, y), v_y(x, y), 0)$$

$$\frac{\partial \mathbf{v}}{\partial t} + \overset{\text{Non-linear}}{\mathbf{v} \cdot \nabla \mathbf{v}} - \overset{\text{Diffusion}}{\frac{1}{\text{Re}} \nabla^2 \mathbf{v}} = 0$$

Vorticity $\zeta \equiv \hat{\mathbf{k}} \cdot \nabla \times \mathbf{v}$



Vorticity equation

$$\frac{\partial \zeta}{\partial t} + \mathbf{v} \cdot \nabla \zeta - \frac{1}{\text{Re}} \nabla^2 \zeta = 0$$



Material
advection

$$\frac{D\zeta}{Dt} = \frac{\nabla^2 \zeta}{\text{Re}}$$

Diffusion

Vortices in 3D flows

Navier-Stokes

$$\frac{\partial \mathbf{u}}{\partial t} + \overbrace{\mathbf{u} \cdot \nabla \mathbf{u}}^{\text{Non-linear}} - \overbrace{\frac{1}{\text{Re}} \nabla^2 \mathbf{u}}^{\text{Diffusion}} = 0$$

Vorticity $\omega \equiv \nabla \times \mathbf{u}$

Vorticity equation

$$\frac{\partial \omega}{\partial t} + \mathbf{u} \cdot \nabla \omega - \frac{1}{\text{Re}} \nabla^2 \omega = (\omega \cdot \nabla) \mathbf{u}$$

3D vortices stretch!

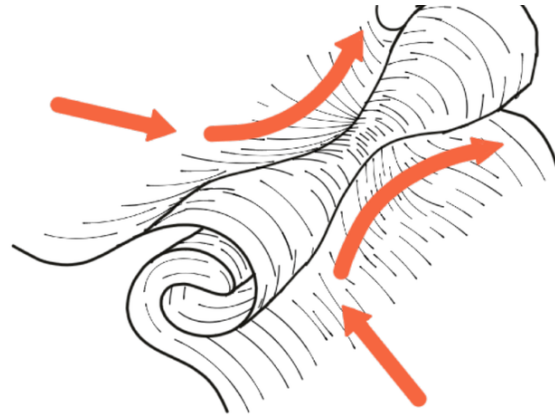
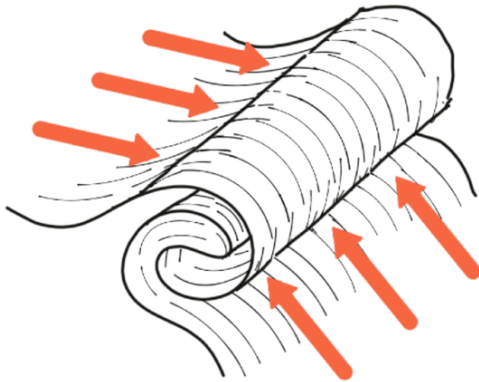
2D vorticity equation

$$\frac{\partial \zeta}{\partial t} + \mathbf{v} \cdot \nabla \zeta - \frac{\nabla^2 \zeta}{\text{Re}} = 0$$

3D Vorticity equation

$$\frac{\partial \omega}{\partial t} + \mathbf{u} \cdot \nabla \omega - \frac{\nabla^2 \omega}{\text{Re}} = (\omega \cdot \nabla) \mathbf{u}$$

Missing term:
Vortex stretching



Finite-time blowup of Navier-Stokes equations

Definition:

Does an incompressible viscous 3D fluid, started from any smooth initial velocity of finite energy, keep flowing smoothly forever?

Or can it concentrate energy onto small scales fast enough to outrun viscous damping and blow the velocity up to infinity in finite time?

Physical interpretation:

In 3D, nonlinear vortex stretching pushes energy to small scales while viscosity smooths it out. We do not know which of the two wins.

$$\frac{\nabla^2 \omega}{\text{Re}} \quad \text{vs.} \quad (\omega \cdot \nabla) \mathbf{u}$$

Viscosity Vortex stretching

Millennium Problem by Clay Institute

Exact mathematical definition from Fefferman's statement:

**Governing equations, for unknown velocity $\mathbf{u}(x, t) \in \mathbb{R}^n$
and pressure $p(x, t) \in \mathbb{R}$, $x \in \mathbb{R}^n$, $t \geq 0$:**

$$(1) \quad \partial_t u_i + \sum_{j=1}^n u_j \frac{\partial u_i}{\partial x_j} = \nu \Delta u_i - \frac{\partial p}{\partial x_i} + f_i(x, t)$$

$$(2) \quad \operatorname{div} \mathbf{u} = \sum_{i=1}^n \frac{\partial u_i}{\partial x_i} = 0$$

$$(3) \quad \mathbf{u}(x, 0) = \mathbf{u}^\circ(x)$$

Admissibility of the data and decay conditions:

$$(4) \quad |\partial_x^\alpha \mathbf{u}^\circ(x)| \leq C_{\alpha K} (1 + |x|)^{-K} \quad \text{for any } \alpha, K$$

$$(5) \quad |\partial_x^\alpha \partial_t^m f(x, t)| \leq C_{\alpha m K} (1 + |x| + t)^{-K} \quad \text{for any } \alpha, m, K$$

What counts as a physically reasonable solution:

$$(6) \quad p, \mathbf{u} \in C^\infty(\mathbb{R}^n \times [0, \infty))$$

$$(7) \quad \int_{\mathbb{R}^n} |\mathbf{u}(x, t)|^2 dx < C \quad \text{for all } t \geq 0$$

1. NS mom. Eq

2. NS div. Eq

3. Initial velocity

4 & 5: Force and initial velocity and their derivs.
must decay faster than powerlaws so no energy
can be pumped in from infinity or forever

6. Smoothness

7. Energy is bounded



Charles Fefferman (1949-)
Princeton University
Fields Medal (mathematical
analysis)

Different solution types

No forcing ($\mathbf{f}_{\text{ext}} = 0$)

Forcing allowed ($\mathbf{f}_{\text{ext}} \neq 0$)

(A) Existence and smoothness on $\mathbb{R}^3$

Let $u^\circ(x)$ be any smooth, divergence-free vector field. Take $f(x, t)$ to be identically zero. Then there exist smooth functions $p(x, t)$, $u_i(x, t)$ on $\mathbb{R}^3 \times [0, \infty)$ that satisfy the equations.

(C) Breakdown on $\mathbb{R}^3$

Then there exist a smooth, divergence-free vector field $u^\circ(x)$ on $\mathbb{R}^3$ and a smooth $f(x, t)$ on $\mathbb{R}^3 \times [0, \infty)$, satisfying initial conditions for which there exist no solutions (p, u) of dynamical equations on $\mathbb{R}^3 \times [0, \infty)$.

(B) Existence and smoothness on $\mathbb{R}^3/\mathbb{Z}^3$

Let $u^\circ(x)$ be any smooth, divergence-free vector field; we take $f(x, t)$ to be identically zero. Then there exist smooth functions $p(x, t)$, $u_i(x, t)$ on $\mathbb{R}^3 \times [0, \infty)$ that satisfy the equations.

(D) Breakdown on $\mathbb{R}^3/\mathbb{Z}^3$

Then there exist a smooth, divergence-free vector field $u^\circ(x)$ on $\mathbb{R}^3$ and a smooth $f(x, t)$ on $\mathbb{R}^3 \times [0, \infty)$, satisfying initial conditions, for which there exist no solutions (p, u) of dynamical equations on $\mathbb{R}^3 \times [0, \infty)$.

Open domain
 $\mathbb{R}^3$

Closed torus
 $\mathbb{T}^3$

Different solution types

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+ (via a corollary)

(D) Breakdown on $\mathbb{R}^3/\mathbb{Z}^3$

Then there exist a smooth, divergence-free vector field $u^\circ(x)$ on $\mathbb{R}^3$ and a smooth $f(x, t)$ on $\mathbb{R}^3 \times [0, \infty)$, satisfying initial conditions, for which there exist no solutions (p, u) of dynamical equations on $\mathbb{R}^3 \times [0, \infty)$.

Note 1: It was historically speculated that there is no blowup so the conditions were relaxed to include forcing. Fefferman states: “To give reasonable leeway to solvers while retaining the heart of the problem, we ask for a proof of one of the following four statements.”

Note 2: Inclusion of periodic boundaries (torus) also helps practitioners avoid possible problems at infinity so they can focus on local blowup.

Open domain
 $\mathbb{R}^3$

Closed torus
 $\mathbb{T}^3$

Timeline of discovery

Foundations

- 1757 — Euler: the inviscid equations
- 1822 — Navier: adds the viscous term
- 1845 — Stokes: the modern form

Classical theory

- 1934 — Leray: global weak solutions with an energy inequality; smoothness left open
- 1958 — Ladyzhenskaya (Lions–Prodi 1959): the 2D case is settled
- 1982 — Caffarelli–Kohn–Nirenberg: partial regularity, singular set of measure zero
- 2003 — Escauriaza–Seregin–Šverák: regularity under a bounded scale-invariant L^3 norm

The prize

- 2000 — Clay Institute, Paris: Millennium Prize; Fefferman states alternatives (A)–(D)

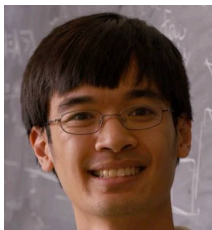
The modern assault

- 2016 — Tao: blowup for an averaged Navier–Stokes
- 2017 — Daneri–Székelyhidi: oscillations realise a prescribed stress (Euler)
- 2019 — Buckmaster–Vicol: nonuniqueness of weak NS solutions
- 2022 — Albritton–Brué–Colombo: non-uniqueness, forced NS, force singular $t = 0$
- 2023 — Córdoba–Martínez-Zorúa: forced 3D Euler blowup, non-smooth force
- 2024 — Córdoba–Martínez-Zorúa–Zheng: hypodissipative NS, rough force
- 2025 — Wang et al: numerical singularities for IPM and Boussinesq

August - September 2026

- 15 Aug — Alpöge & Buckmaster: smooth-forcing blowup for Euler/IPM/Boussinesq
- 1 Sep — OpenAI launches large-scale work on the problem
- 2-6 Sep — Buckmaster, Alpöge and OpenAI exchange emails
- 7 Sep — Mastodon post by Buckmaster (& Alpöge) revealing their Euler solution
- 8 Sep — OpenAI: 166pp with Lean formalization, claiming (C) and (D) of NS

Tao



Córdoba & Martínez-Zorúa



Buckmaster & Alpöge

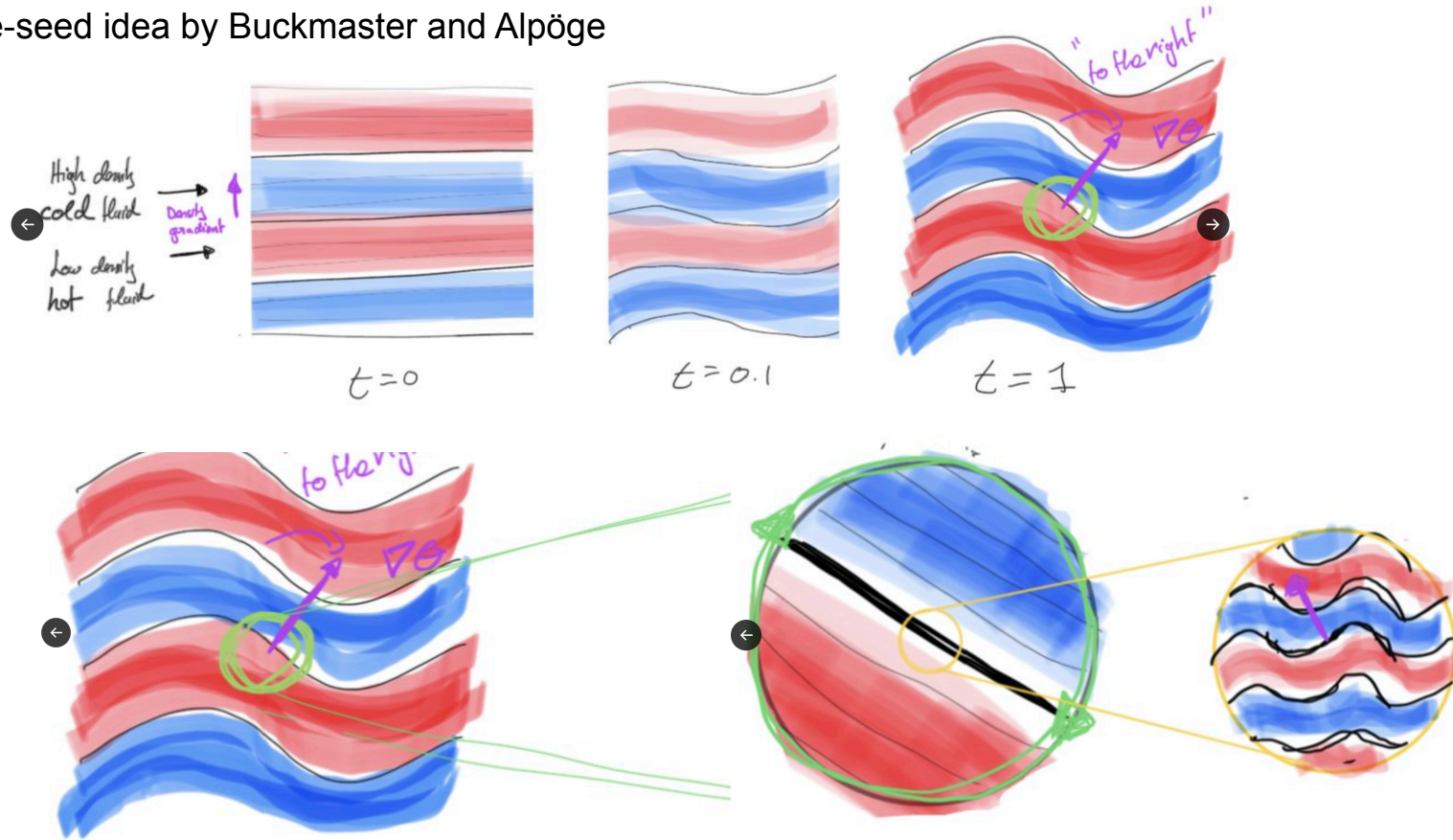


OpenAI



Buckmaster & Alpöge's wave stacking

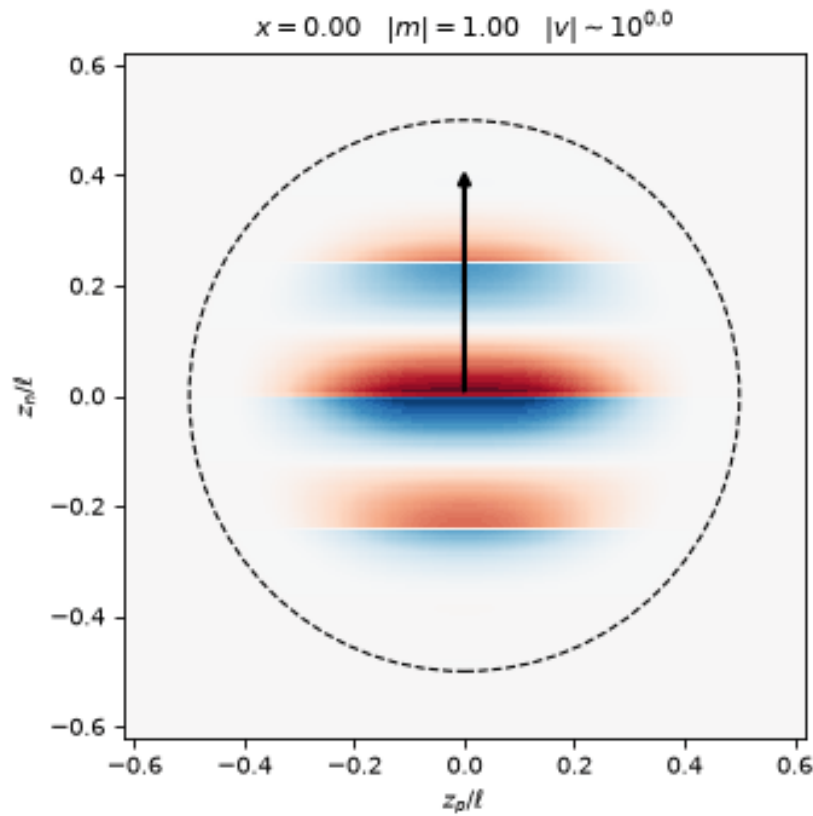
Wave-seed idea by Buckmaster and Alpöge



Finite-time blowup of Euler equation

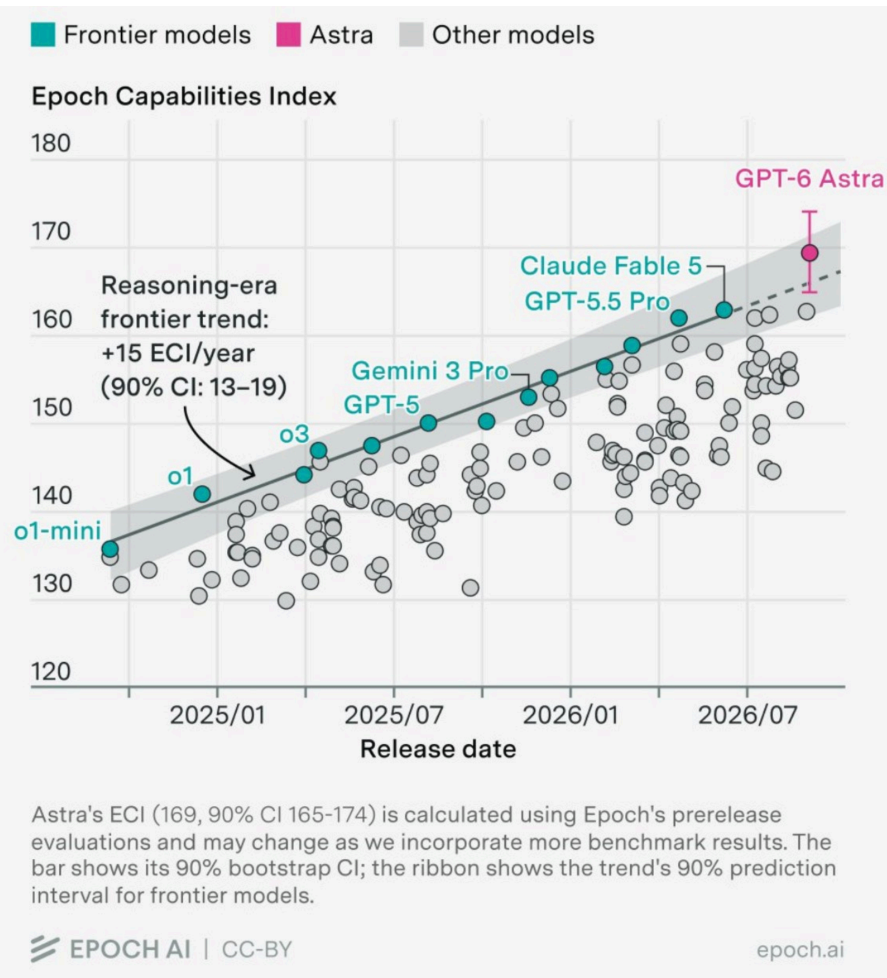
Solution by Buckmaster and Alpöge:

1. A compact ring of swirling fluid
2. Seed a short-wavelength circulation wave; centrifugal (Rayleigh) instability grows it.
3. Return its meridional part to zero; it leaves a steeper angular-momentum gradient.
4. Seed the next, finer wave on that gradient; it grows faster.
5. Vorticity and circulation gradient diverge in finite time; velocity stays bounded.
6. A small force plants each seed and cancels errors, and stays smooth through the singular time.



OpenAI's big push

- 10,000 parallel agents
- 88 hrs of running (1–5 Sep)
- +17 hrs of Lean checking
- Launched on rumours that Anthropic had made progress on Millennium problem
- Compute cost: estimates \$10–30M
- Next-generation internal model, “significantly more capable than GPT-6 Astra”
- ~400 working years in one weekend



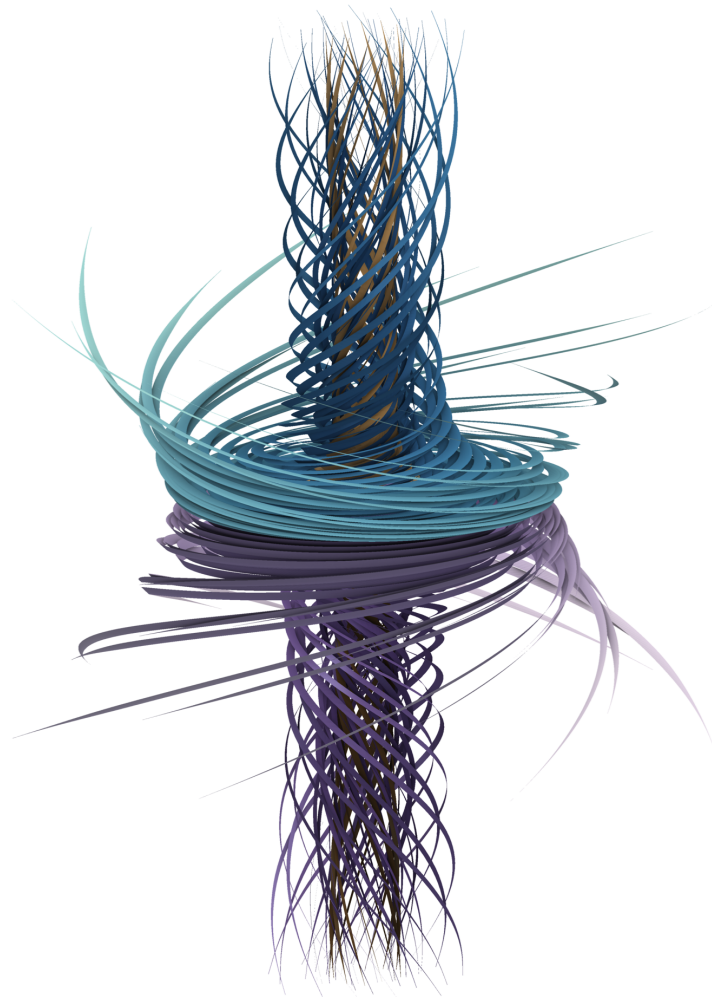
Solution strategy

General form — the force is defined as the residual

$$f \equiv \mathcal{R}(u, p) = \partial_t u + (u \cdot \nabla)u - \Delta u + \nabla p$$

Find a flow that blows up while its residual force stays smooth, bounded and compactly supported.

Results in toroidal flow (blue + purple) and axial flow (brown)



Time evolution of the solution

Axial out/in and azimuthal flow for a small parameter $h \ll 1$
and time to singularity $\tau \equiv 1 - t$:

$$|u_\phi|, |u_z| \asymp \tau^{-1/2-h} \rightarrow \infty, \quad |u_r| = O(\tau^{-1/2}) \rightarrow \infty$$

Note: $a \asymp b \rightarrow c \cdot b \leq a \leq C \cdot b$

Results in **shrinking**

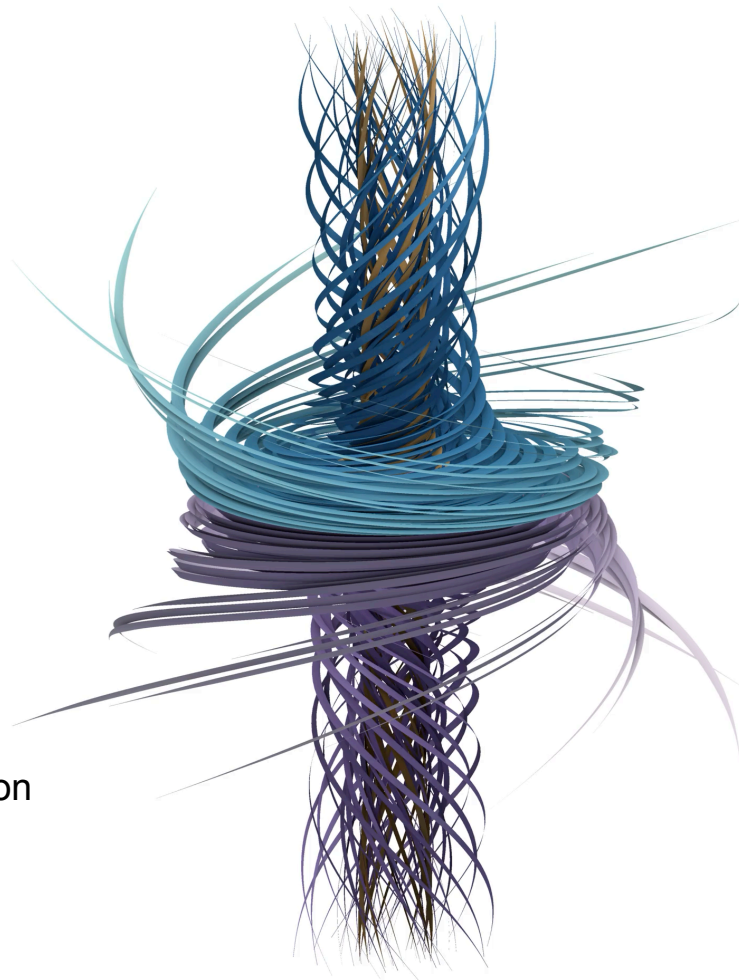
$$\ell_r \asymp \tau^{1/2}, \quad \ell_z \asymp \tau^{1/2-h}$$

And **aspect ratio** becoming (tiny bit) more **slender**

$$\ell_r / \ell_z \asymp \tau^h \rightarrow 0$$

Finite time blowup for the toroidal velocity with a power law solution

$$u_\phi = \tau^{-h-\frac{1}{2}} (\text{const.} + O(\tau^{2h})) \rightarrow \infty$$



Blowup: stretching wins over dissipation

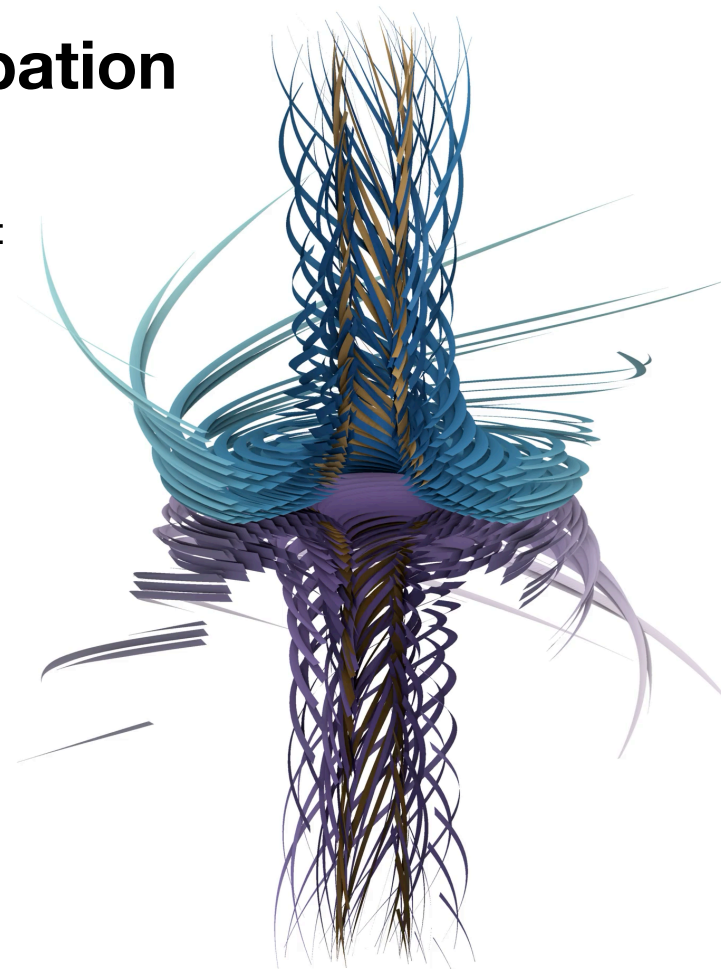
“Fluid makes increasingly many turns during a radial diffusion time”:

$$\frac{\text{swirl turnover}}{\text{radial diffusion}} = \frac{|u_\phi|/\ell_r}{\nu/\ell_r^2} = \text{Re}_\phi \asymp \tau^{-h} \rightarrow \infty$$

$$\text{Core volume} \asymp \tau^{\frac{3}{2}-h}$$

$$\text{Core energy} \asymp \tau^{\frac{1}{2}-3h} \rightarrow 0$$

$$\text{Dissipation rate} \asymp \tau^{-\frac{1}{2}-3h}$$

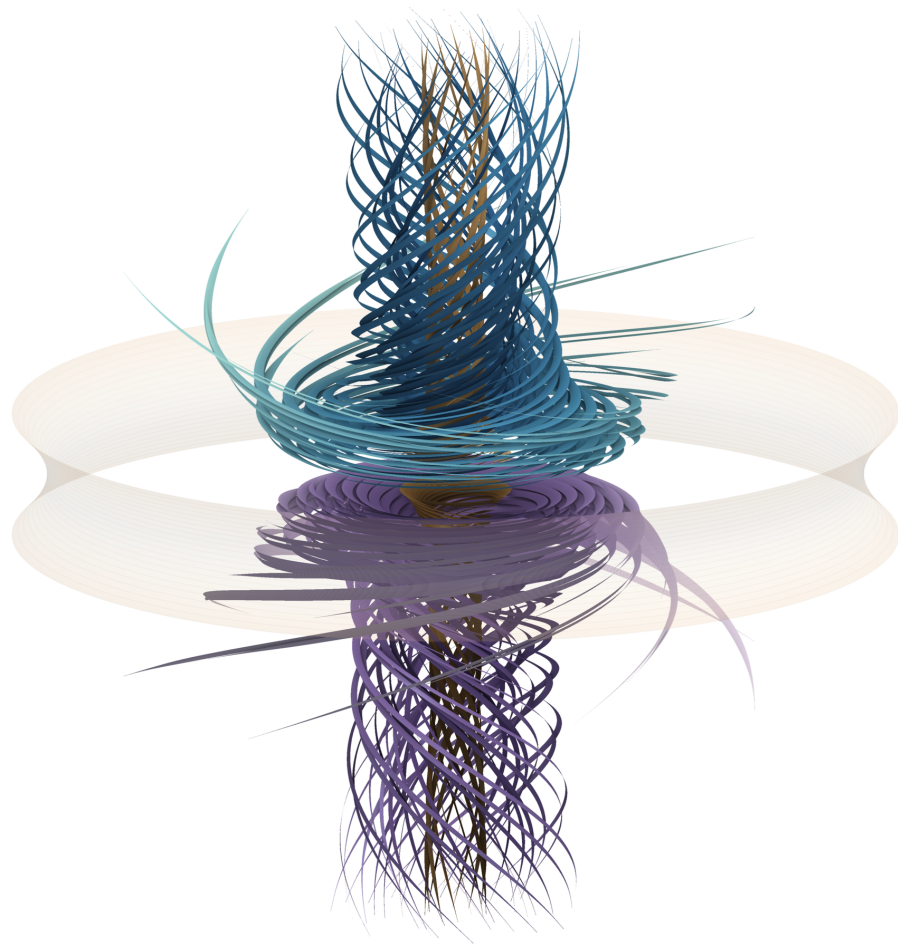
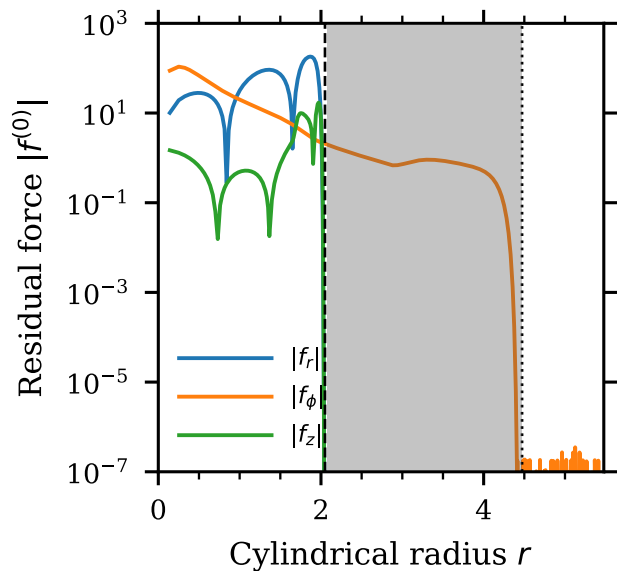


Crux: smooth force

The core needs to smoothly connect to the purely azimuthal, height-independent exterior flow

$$\mathbf{u}_{\text{ext}} = u_\phi(r, t) \mathbf{e}_\phi \text{ (exact radial heat solution).}$$

Imbalance is left in the annulus in between them.



Waves as the “glue”

Oscillatory, ring-shaped pulses w supply the momentum flux the background force cannot without diverging:

$$\mathbf{w} = \mathbf{a} \cos(\mathbf{k} \cdot \mathbf{x} + \varphi), \quad \mathbf{a} \cdot \mathbf{k} = 0, \quad \langle \mathbf{w} \otimes \mathbf{w} \rangle = \frac{1}{2} \mathbf{a} \otimes \mathbf{a}$$

With a force response

$$\mathcal{R}(u_B + w, p_B + \pi) = \underbrace{\mathcal{R}(u_B, p_B)}_{\text{Diverging background}} + L_{u_B}(w, \pi) + \underbrace{\nabla \cdot (w \otimes w)}_{\text{Reynolds stress; balance term}}$$

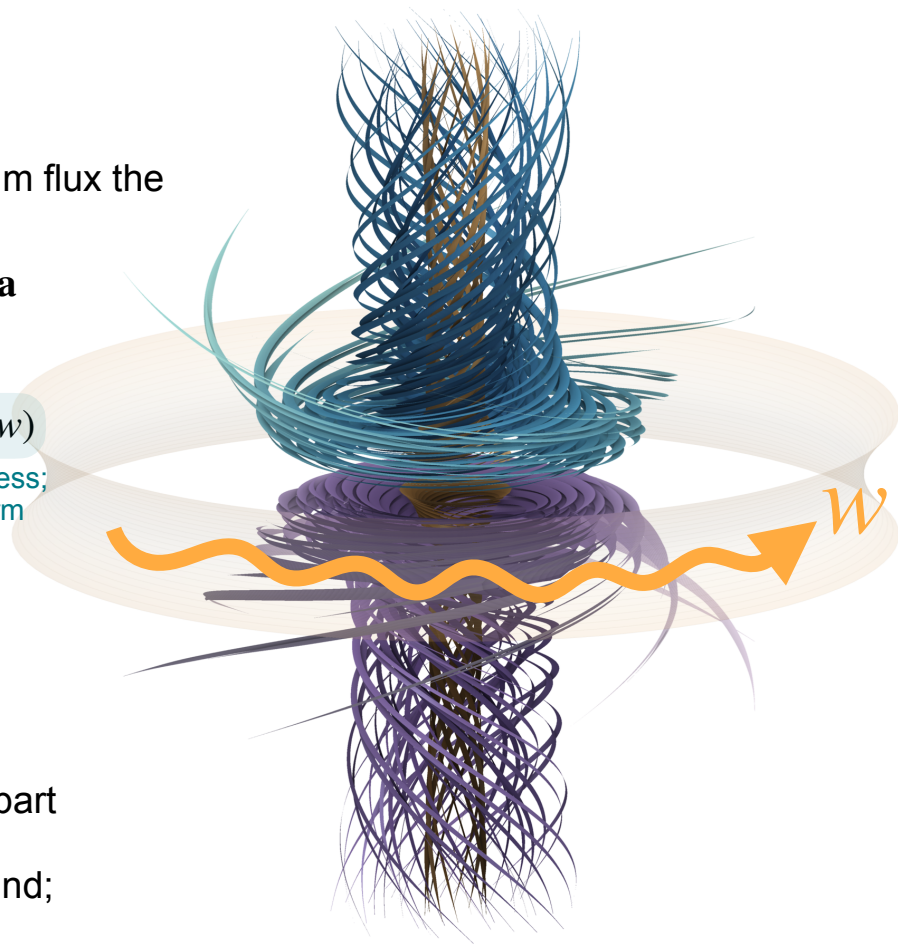
where the linear operator describes the wave evolution

$$L_{u_B}(w, \pi) := \partial_t w + (u_B \cdot \nabla) w + (w \cdot \nabla) u_B - \Delta w + \nabla \pi$$

Design strategy

- Choose the pulses so their flux cancels the singular part of the background residual.
- That quadratic term acts nonlinearly on the background;
- the linear term only transports and amplifies.

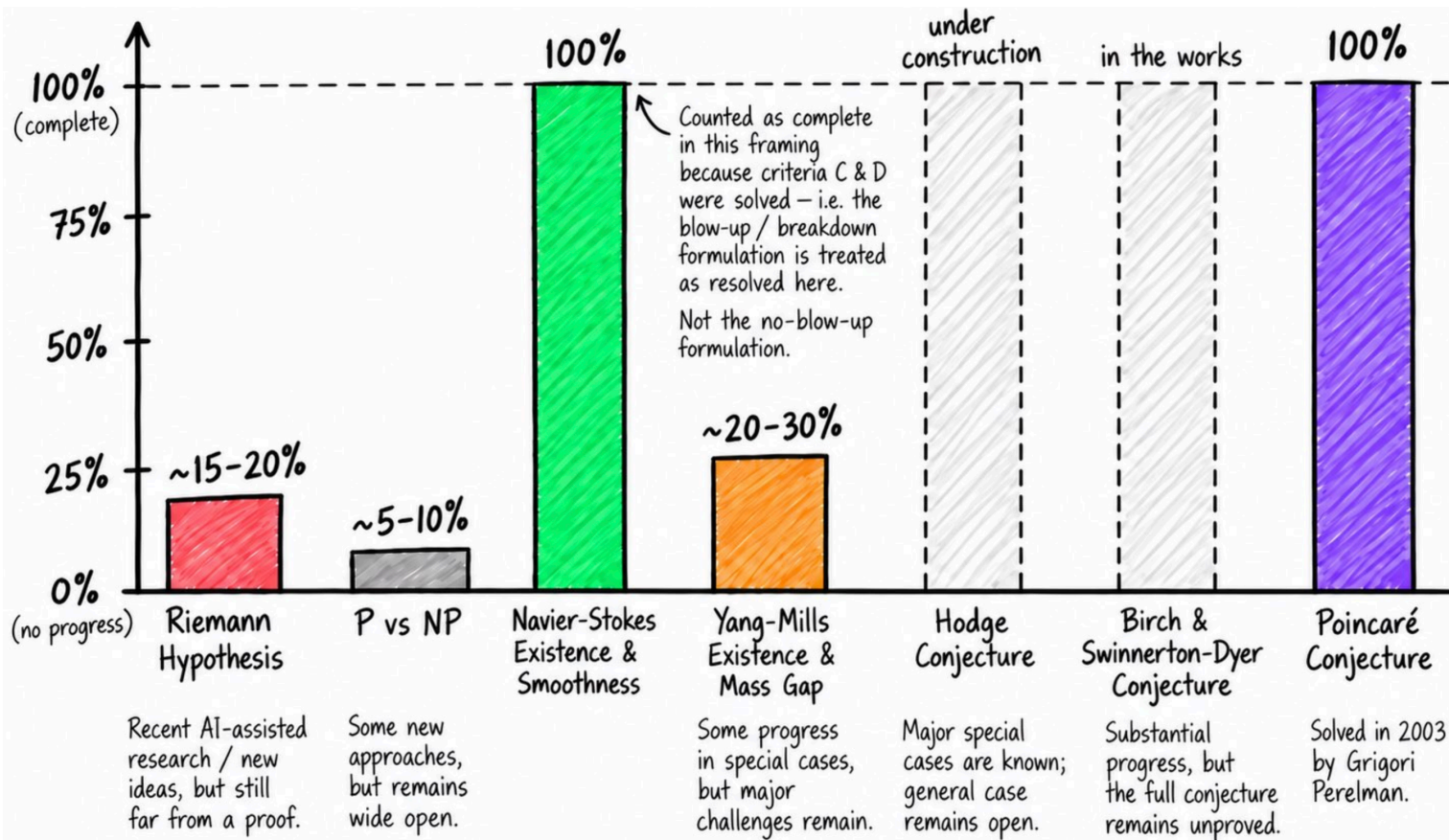
Task: prove this solves the system.



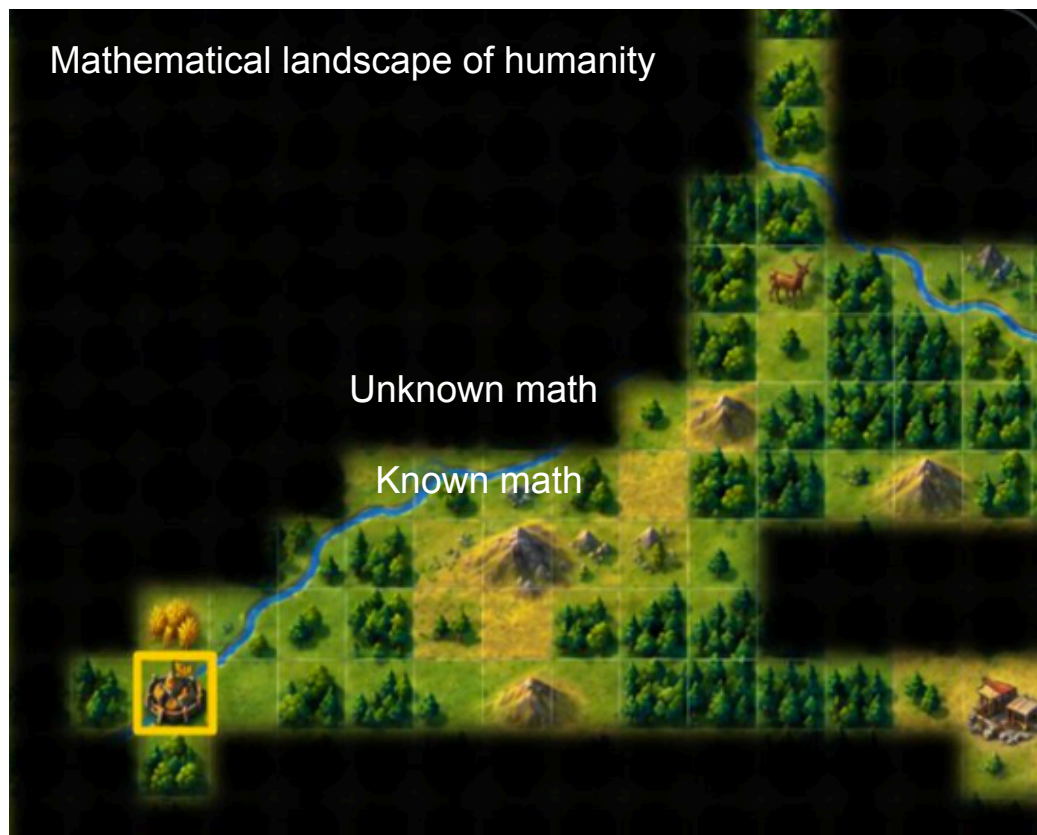
Conclusions

- AI has solved an Millennium problem!
- **Navier-Stokes—thought not to blow up—does blow up**
 - a marginally supercritical, anisotropic vortex can be sustained at the viscous scale
- **Smoothness is obtained with a stabilization by the oscillatory pulses**
 - Idea of seeded, shear-amplified, tilted, material-transported oscillations in the layer resembles Buckmaster & Alpöge (Euler 2026) solution
- **Solution given in the special case with external forcing f_{ext}**
 - Idea by Córdoba & Martínez-Zorúa (Euler 2023; IPM 2024; hypo-dissipative NS 2024)
 - Blowup is constructed, not shown to be attracting (solution type C+D)
- **What did we learn about PDEs:**
 - Alpöge–Buckmaster: oscillations transported by the flow and amplified by instability can be stacked in infinitely many layers while keeping space-time derivatives bounded.
 - OpenAI: wave stress can serve as the closure of a mean flow with viscosity providing the cutoff.

Other Millennium Problems



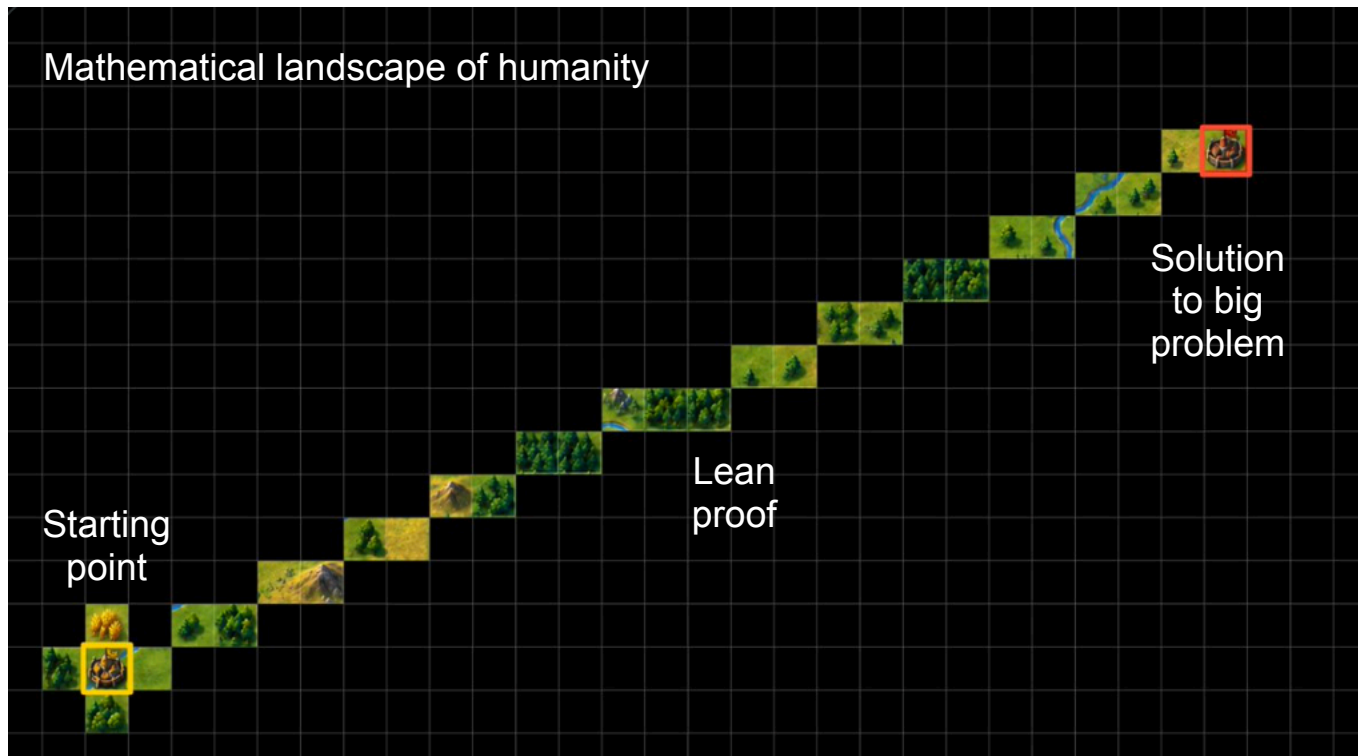
Future of mathematics



Future of mathematics: towards solution



Future of mathematics: AI-assisted Lean proof



Future of mathematics: **artisanal** proof

