

Basics of bound states in field theory

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HIP seminar 6 October 2026

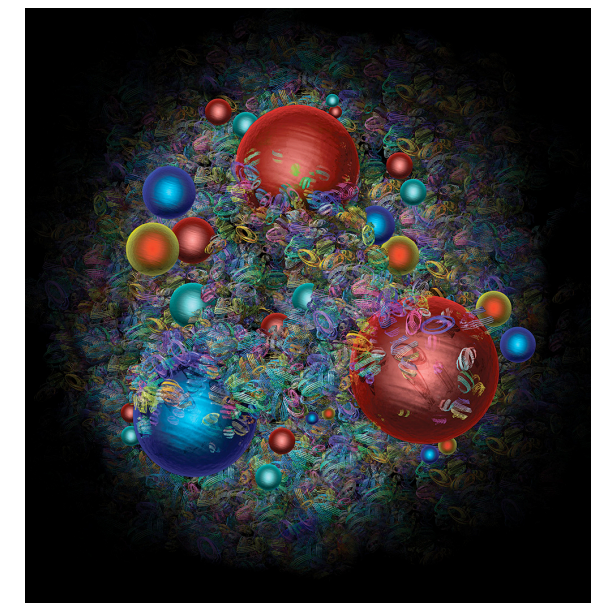
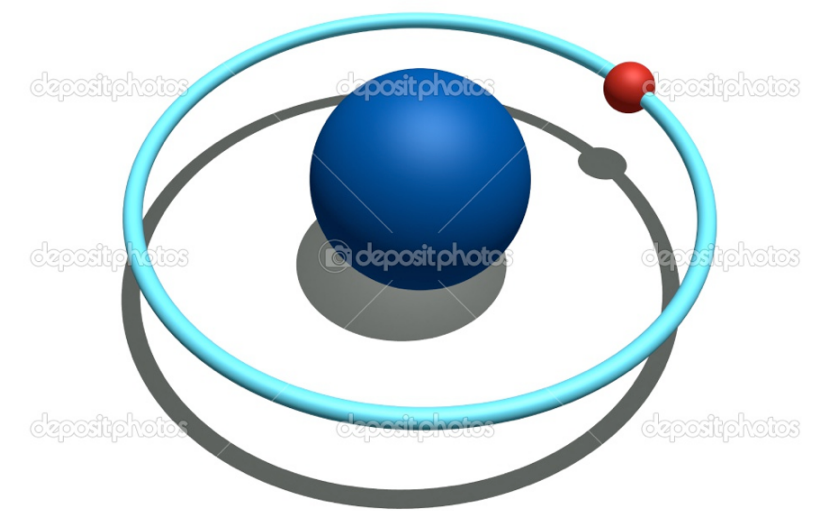
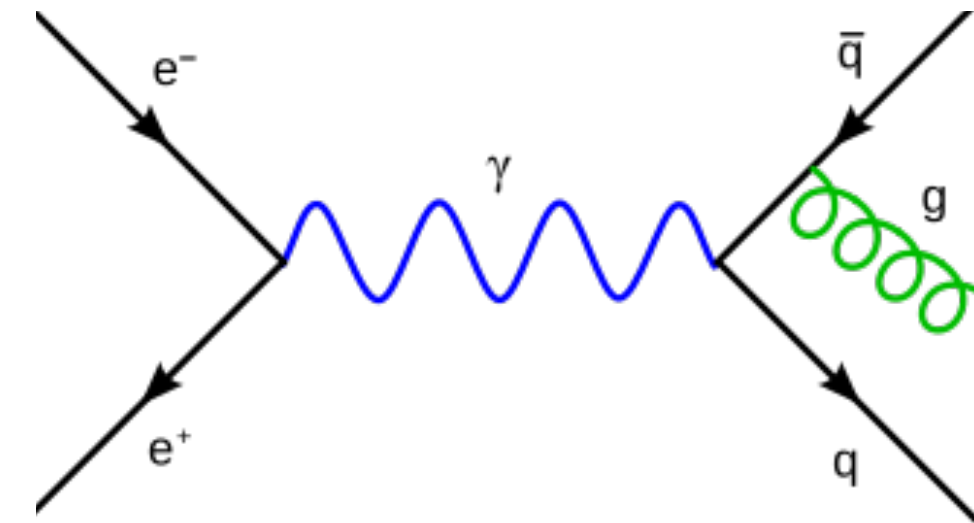
Bound states are the building blocks of matter

Hadrons, atoms, molecules

Analytic methods for QED and QCD are perturbative: $\sum_n c_n \alpha^n$

Scattering amplitudes are expanded around free ($\alpha = 0$) states

Which is the 0th order approximation for bound states?



Positronium (e^+e^-) atoms in QED

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Schrödinger eq. (Quantum Mechanics): $\left(-\frac{\nabla^2}{m_e} - \frac{\alpha}{r} \right) \Phi(x) = E_b \Phi(x)$

Ground state: $\Phi(x) = N \exp(-\alpha r m_e/2)$ (has all powers of α) $E_b = \frac{1}{4}\alpha^2 m_e$

Solution is independent of e^+e^- spin: $S = 1$ (Orthopositronium) vs. $S = 0$ (Parapositronium)

The Schrödinger atom serves as the 0th approximation in QED.

Positronium hyperfine splitting ($S=1$ vs. $S=0$):

$$\frac{\Delta E}{m_e} = \frac{7}{12}\alpha^4 - \left(\frac{8}{9} + \frac{\ln 2}{2}\right)\frac{\alpha^5}{\pi} - \frac{5}{24}\alpha^6 \ln \alpha + \left[\frac{1367}{648} - \frac{5197}{3456}\pi^2 + \left(\frac{221}{144}\pi^2 + \frac{1}{2}\right)\ln 2 - \frac{53}{32}\zeta(3) \right] \frac{\alpha^6}{\pi^2} - \frac{7\alpha^7}{8\pi} \ln^2 \alpha + \left(\frac{17}{3}\ln 2 - \frac{217}{90}\right)\frac{\alpha^7}{\pi} \ln \alpha + \mathcal{O}(\alpha^7)$$

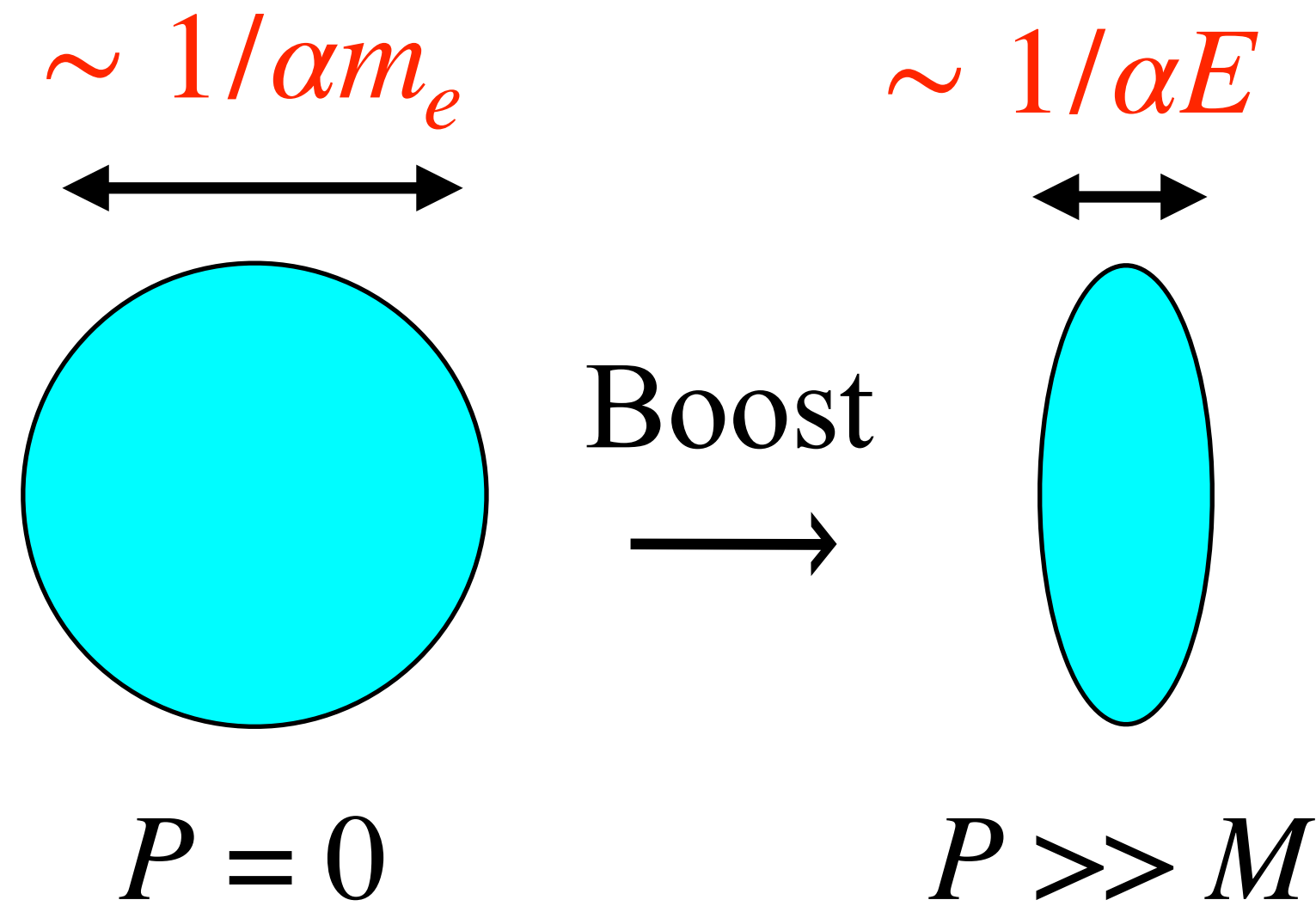
Agrees with experiment: $\Delta\nu_{QED} = 203.39169(41) \text{ GHz}$

$$\Delta\nu_{EXP} = 203.394 \pm .002 \text{ GHz}$$

Bound state review:
Adkins, Cassidy, and Pérez-Ríos (2022)

Positronium in motion

Lorentz contraction of equal-time states:



The Coulomb potential $-\frac{\alpha}{r} \propto P$, whereas excitation energies **decrease** with P :

$$\sqrt{P^2 + (2m + E_B)^2} - \sqrt{P^2 + 4m^2} \simeq \frac{2m E_B}{P}$$

QED: $|e^-e^+\gamma\rangle$ Fock state contributes to E_B at leading $\mathcal{O}(m_e\alpha^2)$ for $P > 0$.

It subtracts the large Coulomb energy, ensuring $E = \sqrt{M^2 + P^2}$

Positronium as poles in $e^+e^- \rightarrow e^+e^-$

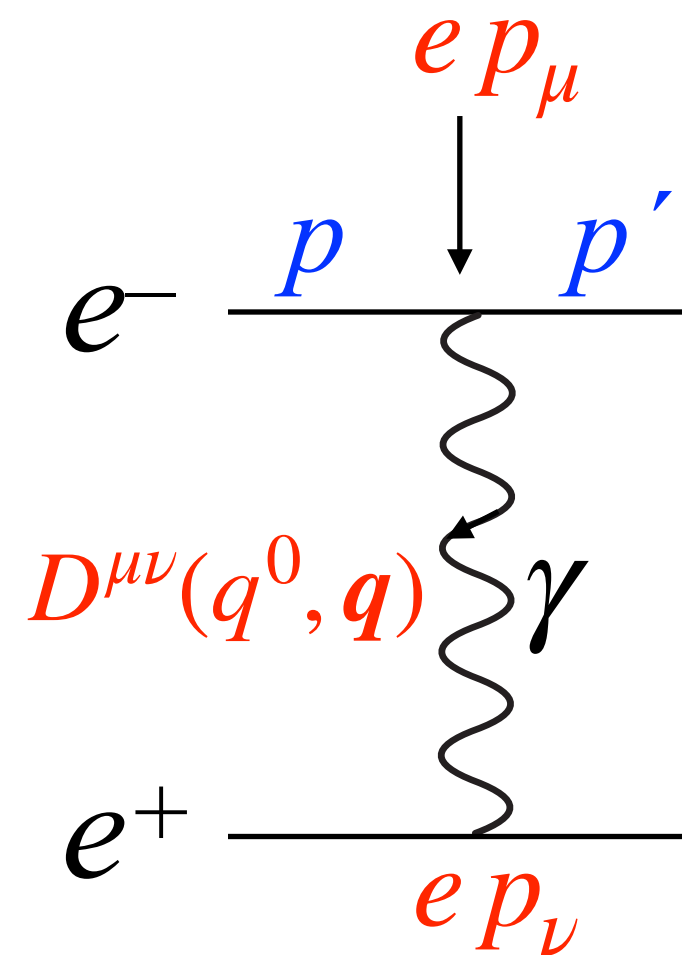
$$= \text{[Ladder diagrams]} + \dots = \frac{R}{(p_1 + p_2)^2 - M^2} + \dots$$

All ladder diagrams are of $O(1/\alpha)$

Rest frame:

$$p_e^0 = \mathcal{O}(m_e) \gg \mathbf{p}_e = \mathcal{O}(\alpha m_e)$$

$$\mathbf{q}_\gamma = \mathcal{O}(\alpha m_e) \gg q_\gamma^0 = \mathcal{O}(\alpha^2 m_e)$$



Born (and all ladders):

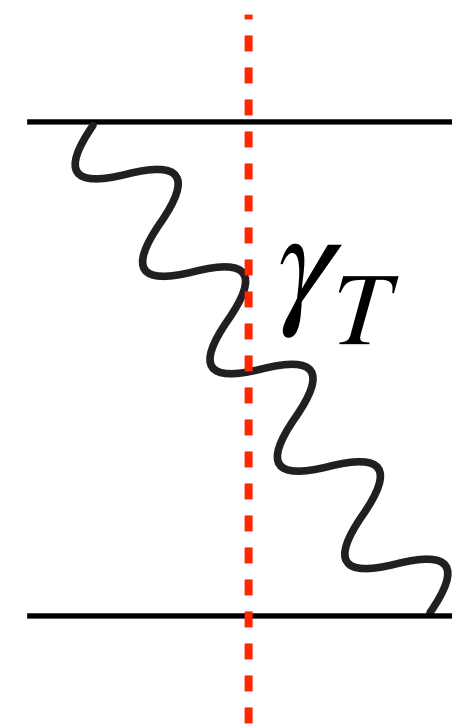
$$\frac{(e p^0)^2}{q^2} \sim \frac{\alpha m_e^2}{\alpha^2 m_e^2} \sim \frac{1}{\alpha}$$

Instantaneous potential

Coulomb ($\mu = 0$)
exchange dominates
in rest frame

Propagating,

transversely
polarized photon
exchange is suppressed



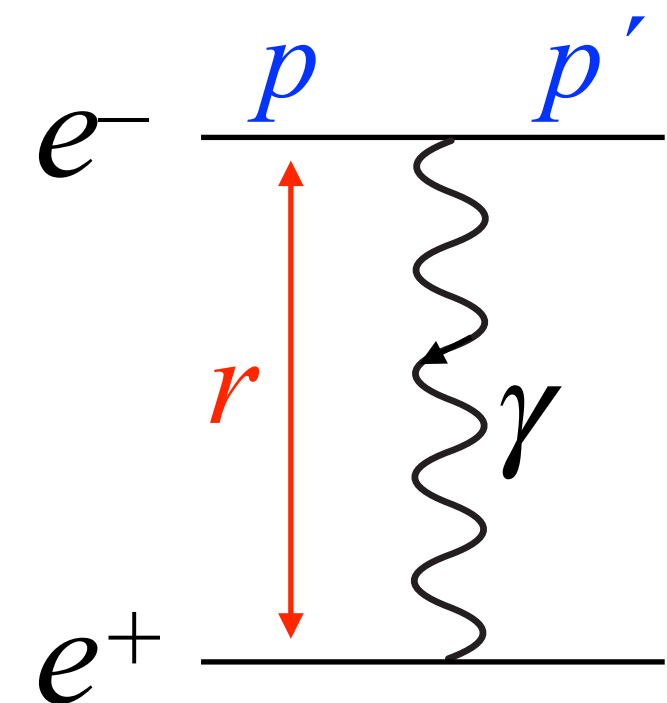
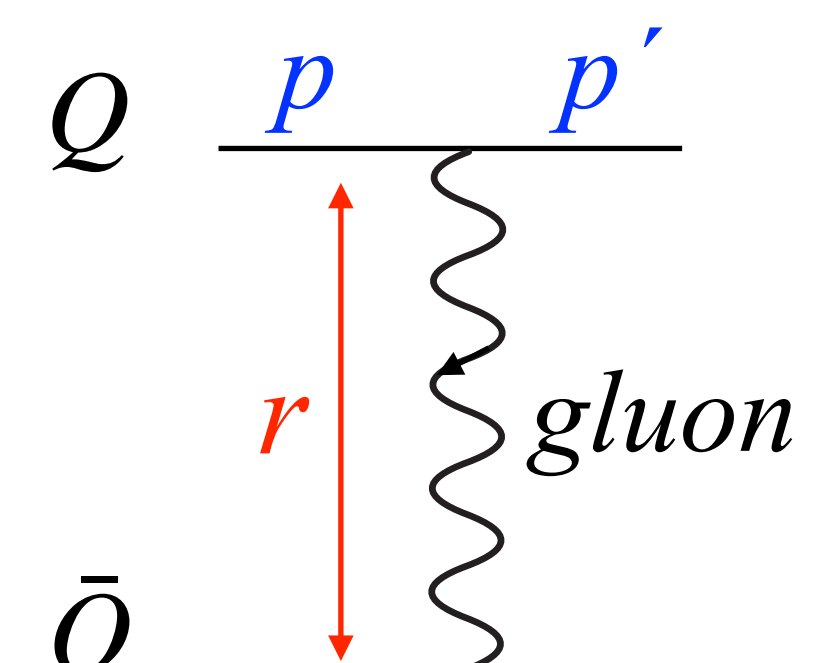
Allows dominance
of e^+e^- Fock state
over $e^+e^-\gamma$ in
Positronium at rest

Requires:

$$\mathbf{p}_e \ll m_e$$

How can strongly bound mesons be $q\bar{q}$, not $q\bar{q}g$?

Schrödinger eq. describes heavy quarkonia

$ e^+e^- \rangle$		$V(r) = -\frac{\alpha}{r}$ Coulomb potential	$\alpha \simeq 1/137$ $r \simeq 1/(\alpha m_e)$	Perturbative QED scale is m_e
$ c\bar{c} \rangle, b\bar{b} \rangle$		$V(r) = V'r - \frac{4}{3} \frac{\alpha_s}{r}$ Cornell potential (1980)	$\alpha_s \simeq 0.39$ $V' \simeq (0.5 \text{ fm})^{-2}$	Perturbative QCD scale is V'

$\frac{\alpha_s}{\pi} \simeq 0.12$ enabled perturbative, $O(\alpha_s^n)$ calculations (hyperfine splitting, decays, ...)

Strong binding is due to the linear potential, α_s is perturbative

Hadrons are classified as $q\bar{q}$ or qqq

$n^{2s+1}\ell_J$	J^{PC}	$ = 1$ $u\bar{d}, \bar{u}d,$ $\frac{1}{\sqrt{2}}(d\bar{d} - u\bar{u})$
1^1S_0	0^{-+}	π
1^3S_1	1^{--}	$\rho(770)$
1^3P_0	0^{++}	$a_0(1450)$
1^1P_1	1^{+-}	$b_1(1235)$
1^3P_1	1^{++}	$a_1(1260)$
1^3P_2	2^{++}	$a_2(1320)$
1^3D_1	1^{--}	$\rho(1700)$
1^1D_2	2^{-+}	$\pi_2(1670)$
1^3D_3	3^{--}	$\rho_3(1690)$
1^3F_4	4^{++}	$a_4(1970)$
1^3G_5	5^{--}	$\rho_5(2350)$
2^1S_0	0^{-+}	$\pi(1300)$
2^3S_1	1^{--}	$\rho(1450)$
2^3P_1	1^{++}	$a_1(1640)$
2^3P_2	2^{++}	$a_2(1700)$
2^1D_2	2^{-+}	$\pi_2(1880)$
3^1S_0	0^{-+}	$\pi(1800)$

Strong (relativistic) binding

$$m_u \simeq 2.16 \text{ MeV}$$

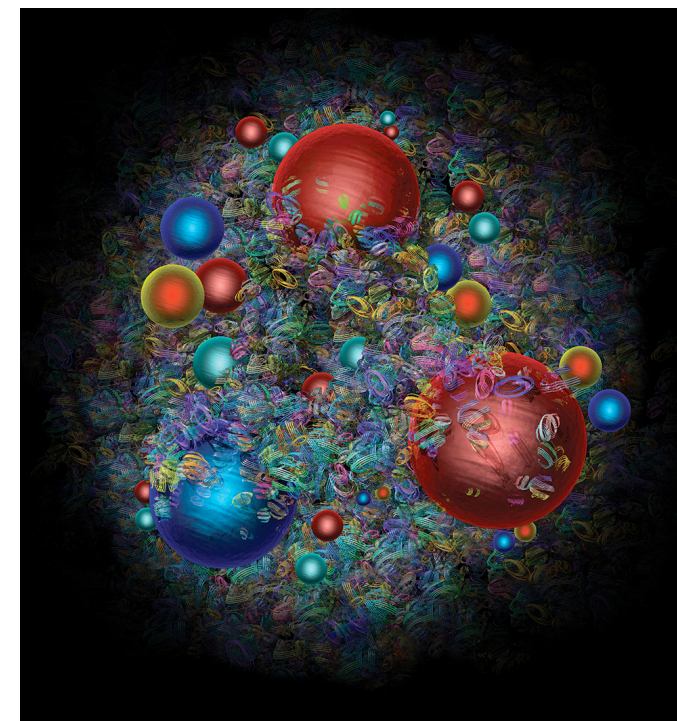
$$m_d \simeq 4.67 \text{ MeV}$$

Mystery:

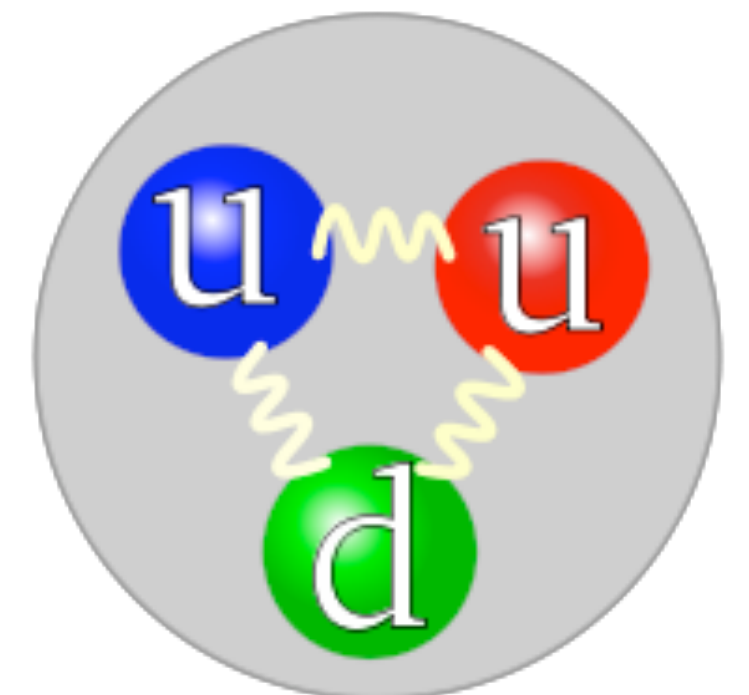
Why does the strong field not create $g, q\bar{q}$?

Hadron quantum numbers reflect only valence quarks.

Expect:



Seen:



The instantaneous potential of gauge theory

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Recall: Electromagnetic waves are transversely polarized. Only two out of four A^μ components propagate.

Kinetic part of gauge boson action:

$$-\frac{1}{4} \int dt d\mathbf{x} F_{\mu\nu} F^{\mu\nu} = \frac{1}{2} \int dt d\mathbf{x} [(\partial_t \mathbf{A} + \nabla A^0)^2 - (\nabla \times \mathbf{A}_T)^2] + \mathcal{O}(A^3, A^4)$$

$\partial_t A^0$ and $\nabla \cdot \mathbf{A}$ are missing

A^0 and $\mathbf{A}_L$ do not propagate

Feynman gauge: $\mathcal{S}_{GF} = -\frac{1}{2} \int dt d\mathbf{x} (\partial_\mu A^\mu)^2 = -\frac{1}{2} \int dt d\mathbf{x} (\partial_t A^0 + \nabla \cdot \mathbf{A})^2$

All four A^μ components propagate (covariant gauge)

An instantaneous (Coulomb) potential binds the e^+e^- Fock state of Positronium

Coulomb gauge: $\nabla \cdot \mathbf{A}(t, \mathbf{x}) = 0$

Temporal gauge: $A^0(t, \mathbf{x}) = 0$

Gauge conditions imposed at an instant of time, at all $\mathbf{x}$

Implies an instantaneous potential

Allows a relativistic binding of $q\bar{q}$ (without transverse gluons)

GAUGE-INVARIANT FORMULATION OF QUANTUM ELECTRODYNAMICS¹

Can.J.Phys. 33 (1955) 650

By P. A. M. DIRAC

ABSTRACT

Electrodynamics is formulated so as to be manifestly invariant under general gauge transformations, through being built up entirely in terms of gauge-invariant dynamical variables. The quantization of the theory can be carried out by the usual rules and meets with the usual difficulties.

It is found that the gauge-invariant operation of creation of an electron involves the simultaneous creation of an electron and of the Coulomb field around it. The requirement of manifest gauge invariance prevents one from using the concept of an electron separated from its Coulomb field.

Free, gauge dependent electron states are used in Feynman diagrams.

ASYMPTOTIC CONDITIONS AND INFRARED DIVERGENCES IN QUANTUM ELECTRODYNAMICS

P. P. Kulish and L. D. Faddeev

Abstract: A definition which is free of infrared divergences is proposed for the S matrix of a relativistic theory of interacting charged particles. This is achieved by a modification of the asymptotic condition and the introduction of a new space of asymptotic states. This state differs from the Fok space, but is separable and relativistically and gauge invariant. The mass operator has no nonvanishing discrete eigenvalues.

Introduction:

... In textbooks on quantum electrodynamics, the reader must wrestle with infrared divergencies and sum the probabilities of a transition from a given initial state to all final states, which include not only detectable particles but also **an arbitrary number of “soft” photons**. ...

Kulish and Faddeev do not cite Dirac (1955)

Temporal gauge: $A^0(t, \mathbf{x}) = 0$

Dirac's formulation is equivalent to temporal gauge: $A^0(t, \mathbf{x}) = 0$.

$A^0 = 0$ is preserved by time independent gauge transformations, which are generated by Gauss' operator $\nabla \cdot \mathbf{E}_L(t, \mathbf{x}) - e\psi^\dagger\psi(t, \mathbf{x})$.

Complete gauge invariance requires physical states to satisfy Gauss law:

$$\left[\nabla \cdot \mathbf{E}_L(t, \mathbf{x}) - e\psi^\dagger\psi(t, \mathbf{x}) \right] |phys\rangle = 0 \quad \text{Dirac (1955)}$$

$\mathbf{E}_L(t, \mathbf{x})$: Electric field operator conjugate to $\mathbf{A}_L(t, \mathbf{x})$

Reviews of temporal gauge do not cite Dirac (1955)

Physical electrons have an EM field

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$$|e^-; t\rangle_{phys} = \int d\mathbf{x} \psi^\dagger(t, \mathbf{x}) W(\mathbf{A}_L; t, \mathbf{x}) |0\rangle \quad [\nabla \cdot \mathbf{E}(t, \mathbf{x}) - e\psi^\dagger\psi(t, \mathbf{x})] |phys\rangle = 0$$

$$W(\mathbf{A}_L; t, \mathbf{x}) = \exp\left[i \int d\mathbf{y} \mathbf{A}_L(t, \mathbf{y}) \cdot \nabla_{\mathbf{y}} \left(\frac{e}{4\pi|\mathbf{y} - \mathbf{x}|}\right)\right] \quad \text{Dirac (1955)}$$

Positronium atom at rest: Physical $e^-(\mathbf{x}_1)$ and $e^+(\mathbf{x}_2)$ distributed by wave function $\Phi(\mathbf{x}_1 - \mathbf{x}_2)$

$$|e^+e^-, t=0\rangle = \int d\mathbf{x}_1 d\mathbf{x}_2 \bar{\psi}_\alpha(\mathbf{x}_1) \Phi_{\alpha\beta}(\mathbf{x}_1 - \mathbf{x}_2) W[\mathbf{A}_L; \mathbf{x}_1, \mathbf{x}_2] \psi_\beta(\mathbf{x}_2) |0\rangle$$

$$W[\mathbf{A}_L, \mathbf{x}_1, \mathbf{x}_2] = \exp\left[\frac{ie}{4\pi} \int d\mathbf{y} \mathbf{A}_L(\mathbf{y}) \cdot \nabla_{\mathbf{y}} \left(\frac{1}{|\mathbf{y} - \mathbf{x}_1|} - \frac{1}{|\mathbf{y} - \mathbf{x}_2|}\right)\right]$$

Bound state condition: $H_{QED} |e^+e^-\rangle = (2m_e + E_b) |e^+e^-\rangle$

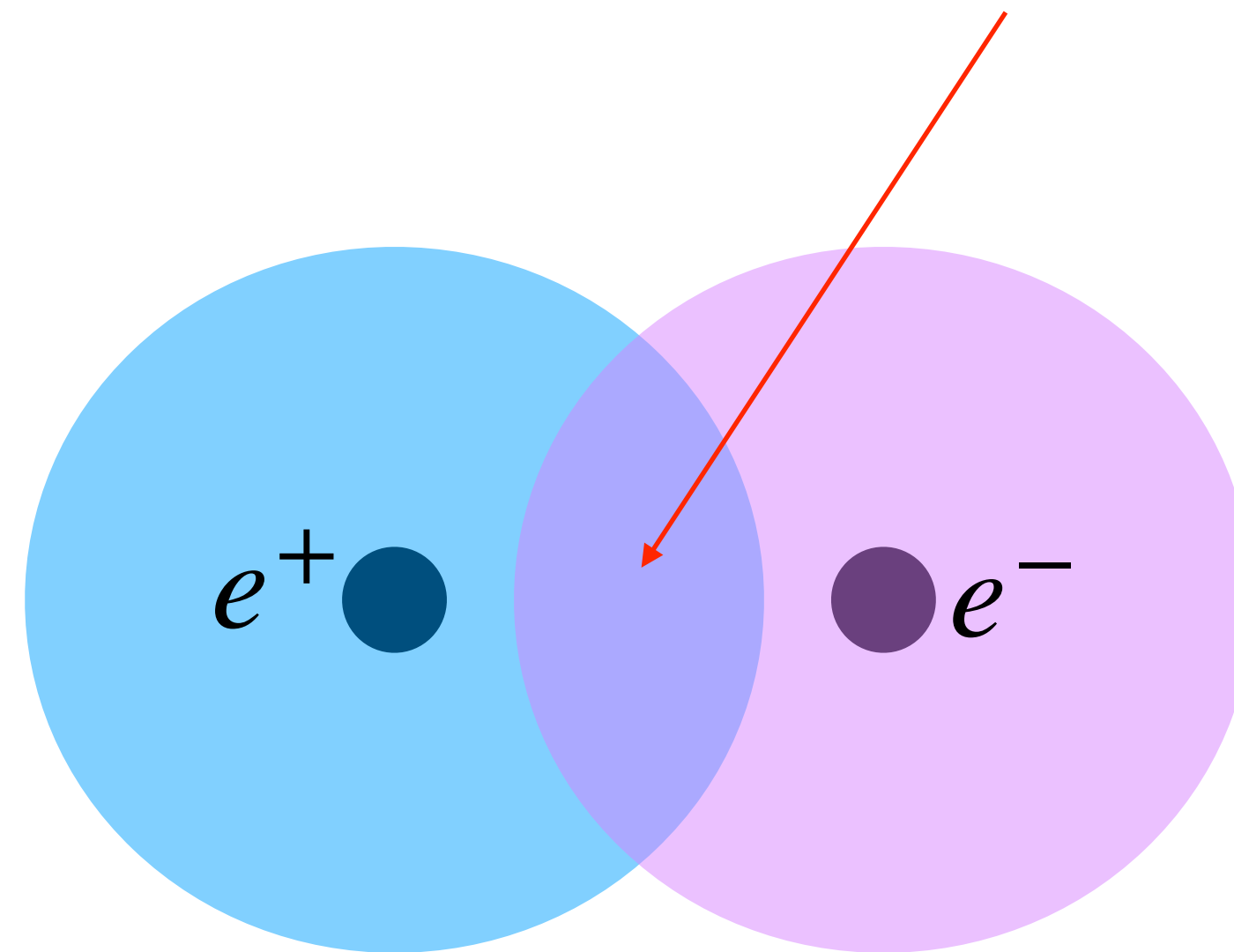
The QED Hamiltonian: $H_{QED} = \int d\mathbf{x} \left[\frac{1}{2}(\mathbf{E}_L^2 + \mathbf{E}_T^2) + \frac{1}{4}F^{ij}F^{ij} + \psi^\dagger(-i\boldsymbol{\alpha} \cdot \nabla + m\gamma^0 - e\boldsymbol{\alpha} \cdot \mathbf{A})\psi \right]$

higher order corrections
↓

Coulomb potential from field interference

$$W[A_L, \mathbf{x}_1, \mathbf{x}_2] = \exp \left[\frac{ie}{4\pi} \int d\mathbf{y} A_L(\mathbf{y}) \cdot \nabla_{\mathbf{y}} \left(\frac{1}{|\mathbf{y} - \mathbf{x}_1|} - \frac{1}{|\mathbf{y} - \mathbf{x}_2|} \right) \right]$$

$$\int d\mathbf{x} \left[\frac{1}{2} E_L^2, W[A_L; \mathbf{x}_1, \mathbf{x}_2] \right] = - \frac{\alpha}{|\mathbf{x}_1 - \mathbf{x}_2|} W[A_L; \mathbf{x}_1, \mathbf{x}_2] \quad (+ \text{ self-energies})$$



Bound states have not been considered in $A^0 = 0$ gauge.

QCD bound states may be analogous to Positronium.

$q\bar{q}$ bound states in QCD, with $A^0 = 0$

$$|q\bar{q}, t = 0\rangle = \int d\mathbf{x}_1 d\mathbf{x}_2 \bar{\psi}_A(\mathbf{x}_1) \Phi(\mathbf{x}_1 - \mathbf{x}_2) W_{AB}[\mathbf{A}_L; \mathbf{x}_1, \mathbf{x}_2] \psi_B(\mathbf{x}_2) |0\rangle$$

Invariance of $|q\bar{q}\rangle$ under time independent gauge transformations requires:

$$\nabla \cdot \mathbf{E}_L^a(\mathbf{x}) |q\bar{q}\rangle = \left[-g f_{abc} \mathbf{A}_T^b \cdot \mathbf{E}_T^c(\mathbf{x}) + g \psi^\dagger T^a \psi(\mathbf{x}) \right] |q\bar{q}\rangle \quad \text{Gauss' constraint}$$

which is ensured by

$$W[\mathbf{A}_L; \mathbf{x}_1, \mathbf{x}_2] \equiv \exp \left[\frac{ig}{4\pi} \int d\mathbf{y} T^a \mathbf{A}_L^a(\mathbf{y}) \cdot \nabla_y \left(\frac{1}{|\mathbf{y} - \mathbf{x}_1|} - \frac{1}{|\mathbf{y} - \mathbf{x}_2|} \right) \right]$$

This gives the

“One-gluon exchange” potential:

$$\int d\mathbf{x} \left[\frac{1}{2} \mathbf{E}_{L,a}^2, W[\mathbf{A}_L; \mathbf{x}_1, \mathbf{x}_2] \right] = -C_F \frac{\alpha_s}{|\mathbf{x}_1 - \mathbf{x}_2|} W[\mathbf{A}_L; \mathbf{x}_1, \mathbf{x}_2]$$

Where does the QCD scale come from?

Scale invariance in QCD

The QCD action is invariant under translations in time and space, implying conserved currents:

$$\partial_\mu T^{\mu\nu}(x) = 0 \quad \text{Energy-Momentum tensor: } T^{\mu\nu} = \frac{1}{2} \bar{\psi} i \overleftrightarrow{D}^{(\mu} \gamma^{\nu)} \psi + \frac{1}{4} g^{\mu\nu} F^2 - F^{\mu\alpha} F^\nu{}_\alpha$$

Four-momentum $P^\nu \equiv \int d\mathbf{x} T^{0\nu}$ is conserved: $\partial_\mu T^{\mu\nu} = 0$ implies $\partial_t P^\nu = 0$

For **massless quarks** the QCD action lacks a scale: It is invariant under **dilations** $x^\mu \rightarrow \lambda x^\mu$

Conservation of the dilation current: $j_D^\mu \equiv x_\nu T^{\mu\nu}$ implies $\partial_\mu j_D^\mu = T^\rho{}_\rho = 0$

But hadrons do have a scale: Their radius is of $O(1 \text{ fm})$, or equivalently $\Lambda_{QCD} \sim 200 \text{ MeV}$

The QCD Trace anomaly

Renormalization breaks the classical scale invariance:

$$T^\alpha_\alpha(x) = \frac{\beta(g)}{2g} F^{a,\mu\nu} F^a_{\mu\nu}(x) \neq 0$$

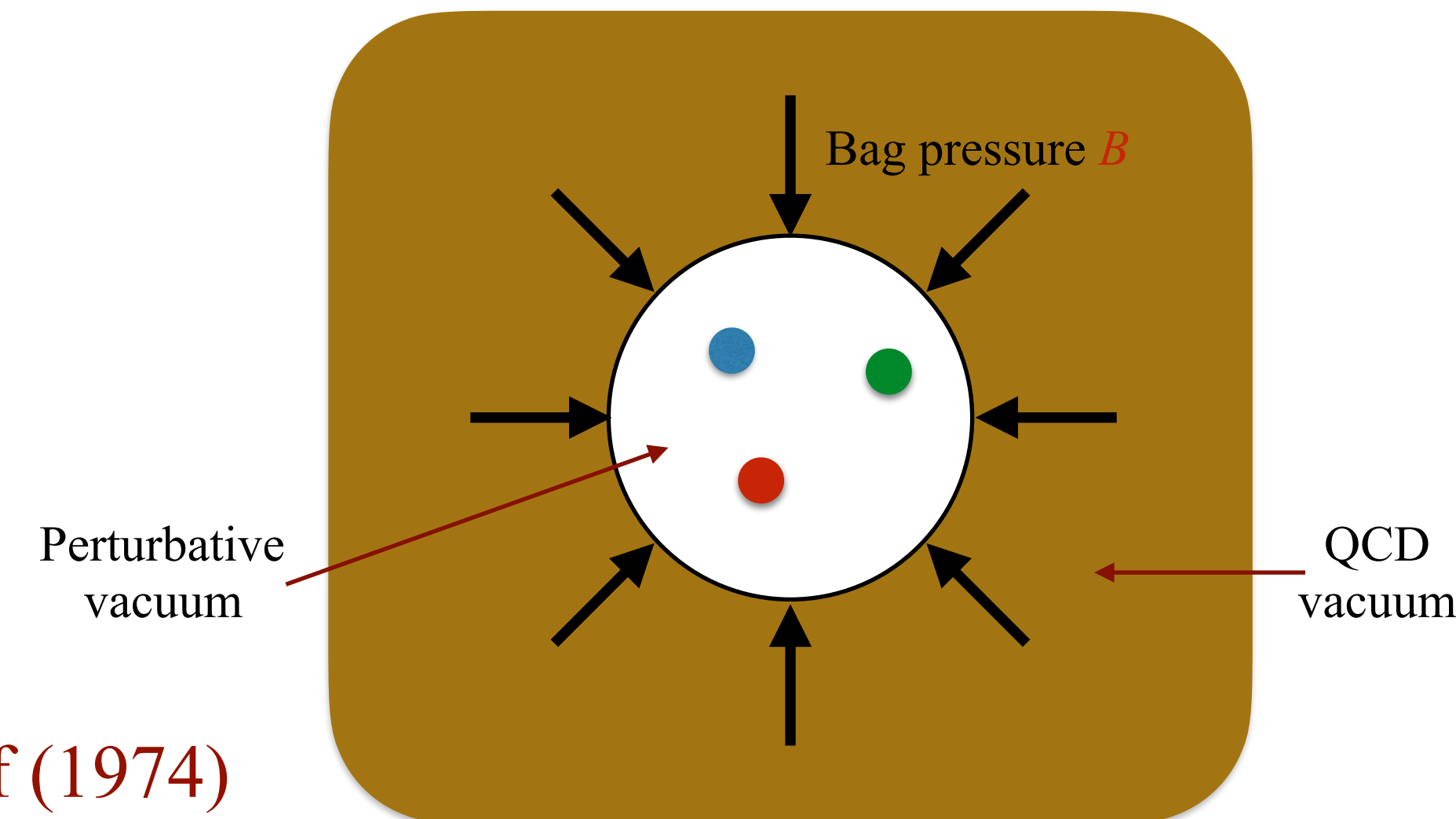
where the QCD β -function is $\beta(g) \equiv \mu \frac{dg(\mu)}{d\mu} = -\frac{\beta_0 g^3}{(4\pi)^2} + \dots$ and $F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu + ig[A_\mu, A_\nu]$

The vacuum expectation value $\langle 0 | -T^\alpha_\alpha(x) | 0 \rangle = \mathcal{E}_\Lambda$ is the energy density of the “gluon condensate”

“The trace anomaly contribution is analogous to the vacuum energy in the MIT bag model. It is explicitly connected to quark confinement.”

Xiangdong Ji (1995)

The MIT model postulates a bag of free quarks inside a vacuum condensate.



Chodos, Jaffe, Johnson, Thorn and Weisskopf (1974)

Adding a homogeneous term to $W[A_L]$

$$W_\kappa[A_L; \mathbf{x}_1, \mathbf{x}_2] \equiv \exp \left\{ i \int_{\mathbf{x}_2}^{\mathbf{x}_1} d\mathbf{y} T^a \mathbf{A}_L^a(\mathbf{y}) \cdot \nabla_{\mathbf{y}} \left[\frac{g}{4\pi} \left(\frac{1}{|\mathbf{y} - \mathbf{x}_1|} - \frac{1}{|\mathbf{y} - \mathbf{x}_2|} \right) + \kappa \mathbf{y} \cdot (\mathbf{x}_1 - \mathbf{x}_2) \right] \right\}$$

Gauss' constraint
is unaffected by the
homogeneous term

$$\nabla_{\mathbf{x}}^2 \left[\frac{1}{|\mathbf{x} - \mathbf{x}_1|} - \frac{1}{|\mathbf{x} - \mathbf{x}_2|} + \kappa \mathbf{x} \cdot (\mathbf{x}_1 - \mathbf{x}_2) \right] = -4\pi [\delta(\mathbf{x} - \mathbf{x}_1) - \delta(\mathbf{x} - \mathbf{x}_2)]$$

$\kappa \mathbf{y} \cdot (\mathbf{x}_1 - \mathbf{x}_2)$ is the only source-less solution which preserves translation and rotation symmetry (for color singlet states).

W_κ gives a universal energy density of the vacuum: $\mathcal{E}_\Lambda = \frac{\Lambda^4}{2g^2 C_F}$. Has a single parameter Λ .

Interference with the Coulomb field gives the confining, instantaneous (Cornell) potential:

$$V(\mathbf{x}_1 - \mathbf{x}_2) = \Lambda^2 |\mathbf{x}_1 - \mathbf{x}_2| - C_F \frac{\alpha_s}{|\mathbf{x}_1 - \mathbf{x}_2|}$$

Summary: Observation

Bound state perturbation theory is not in textbooks.

C.f. the derivation of the perturbative expansion for scattering amplitudes in the Interaction Picture.

THE STATE IS NOT ABOLISHED, IT WITHERS AWAY: HOW QUANTUM FIELD THEORY BECAME A THEORY OF SCATTERING

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Max Planck Institute for the History of Science, Boltzmannstraße 22, 14195
Berlin, Germany

12th November 2020

2011.0598

Learning quantum field theory (QFT) for the first time, after first learning quantum mechanics (QM), one is (or maybe, rather, I was) struck by the change of emphasis: **The notion of the quantum state**, which plays such an essential role in QM, from the stationary states of the Bohr atom, over the Schrödinger equation to the interpretation debates over measurement and collapse, **seems to fade from view when doing QFT.**

Summary: Claim

The belief that $\alpha_s(\Lambda_{QCD}) > 1$ may be incorrect.

Relativistic binding is ensured by confinement (*c.f.* Cornell potential)

$Q^2 \gtrsim \Lambda_{QCD}^2$ is ensured by hadron radii.

Gribov estimated that α_s stops running at $\alpha_s^{crit} = \pi C_F^{-1} (1 - \sqrt{2/3}) \simeq 0.43$

Gribov, hep-ph/9807224 and hep-ph/9902279
Dokshitzer and Kharzeev, hep-ph/0404216

Summary (1)

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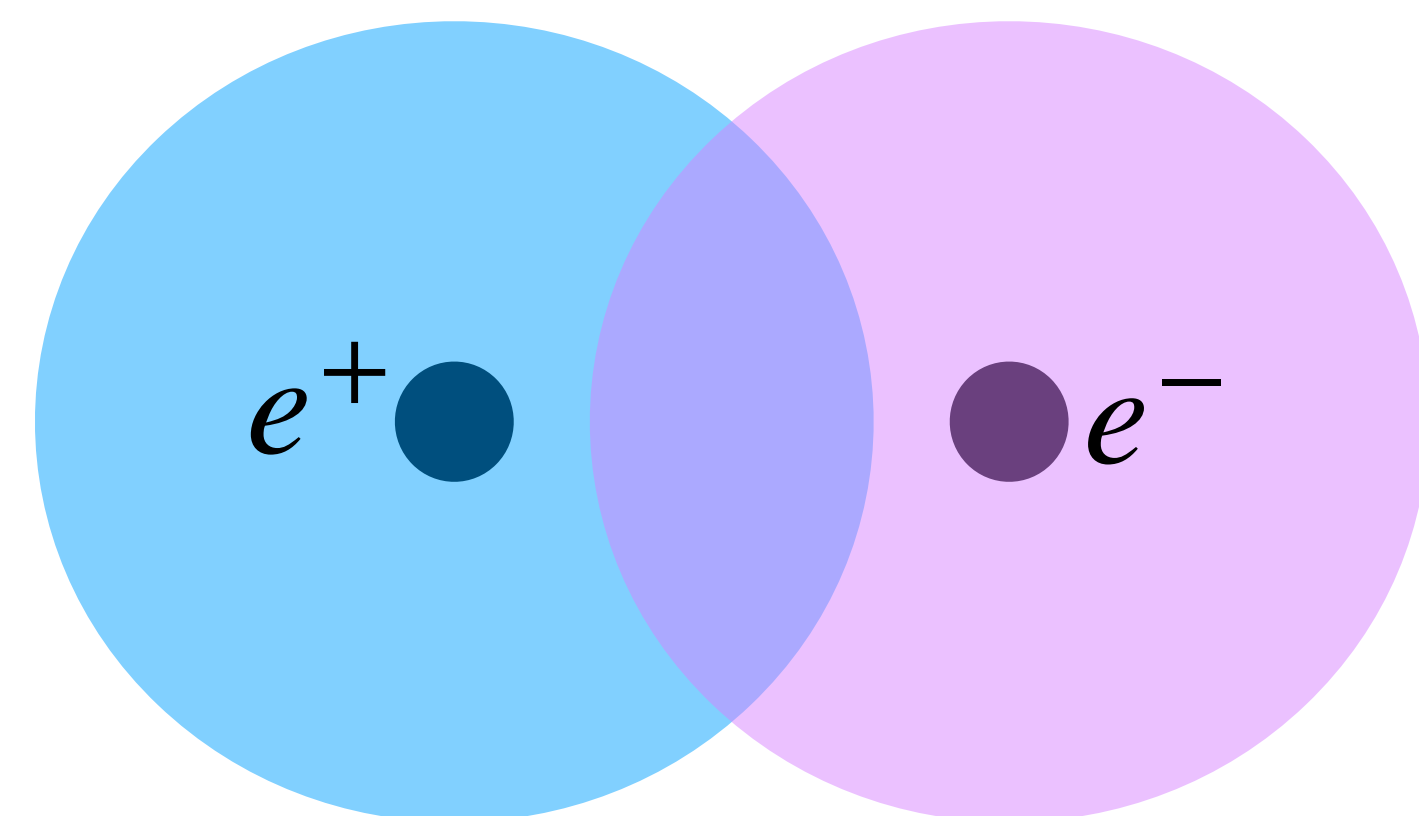
Need to identify a 0th order approximation for a bound state expansion.

Physical electrons have a Coulomb A_L field. Dirac (1955)

“Instantaneous photon exchange” is due to gauge fixing over all space at the same time.

Interference of the A_L fields gives rise to the Coulomb potential

$$-\frac{\alpha}{r} \text{ in QED, and } -C_F \frac{\alpha_s}{r} \text{ in QCD}$$



Summary (2)

The QCD trace anomaly implies an isotropic vacuum energy with scale Λ_{QCD} (*c.f.* MIT bag model).

A homogeneous solution of Gauss' constraint gives a finite gluon energy density $\mathcal{E}_\Lambda$ in space.

Interference with the Coulomb potential causes confining potentials for color singlet states
($q\bar{q}$, qqq , gg , $q\bar{q}g\dots$)

The $q\bar{q}$ meson spectrum looks promising: Linear Regge trajectories: $J = \alpha_0 + \alpha' M^2$

Feel free to ask me about further consequences!

Thank you!

2101.06721
2304.11903
2501.18352
2509.01510
2606.08489

Backup slides

Canonical quantization in $A^0(t, \mathbf{x}) = 0$ gauge

The electric field $\mathbf{E} = -\partial_t \mathbf{A}$ is conjugate to $-\mathbf{A}$. A^0 and its conjugate field vanish

Canonical commutators: $[E^i(t, \mathbf{x}), A^j(t, \mathbf{y})] = i \delta^{ij} \delta(\mathbf{x} - \mathbf{y})$ $\{\psi_\alpha^\dagger(t, \mathbf{x}), \psi_\beta(t, \mathbf{y})\} = \delta_{\alpha\beta} \delta(\mathbf{x} - \mathbf{y})$

Time-independent gauge transformations preserve $A^0 = 0$

$G(t, \mathbf{x}) \equiv \frac{\delta \mathcal{S}_{QED}}{\delta A^0(t, \mathbf{x})} = \nabla \cdot \mathbf{E}(t, \mathbf{x}) - e \psi^\dagger \psi(t, \mathbf{x})$ Gauss' operator $G(t, \mathbf{x})$ generates time-independent gauge transformations

Physical states must be fully gauge fixed: $G(t, \mathbf{x}) |phys\rangle = 0$

An electron state $b^\dagger(\mathbf{p}, \lambda) |0\rangle$ without its Coulomb field is gauge dependent.

Confining potential from $\kappa \mathbf{x} \cdot (\mathbf{x}_1 - \mathbf{x}_2)$

Contribution of $\mathbf{A}_L$ field to Hamiltonian:

$$\begin{aligned} \frac{1}{2} \int d\mathbf{x} [\mathbf{E}_{L,a}^2(\mathbf{x}), W_\kappa[\mathbf{A}_L; \mathbf{x}_1, \mathbf{x}_2]] &= W_\kappa[\mathbf{A}_L; \mathbf{x}_1, \mathbf{x}_2] \frac{1}{2} C_F \int d\mathbf{x} \left[\frac{g}{4\pi} \nabla_x \left(\frac{1}{|\mathbf{x} - \mathbf{x}_1|} - \frac{1}{\mathbf{x} - \mathbf{x}_2} \right) + \kappa (\mathbf{x}_1 - \mathbf{x}_2) \right]^2 \\ &= -C_F \frac{\alpha_s}{|\mathbf{x}_1 - \mathbf{x}_2|} + C_F g \kappa (\mathbf{x}_1 - \mathbf{x}_2)^2 + \frac{1}{2} C_F \kappa^2 (\mathbf{x}_1 - \mathbf{x}_2)^2 \int d\mathbf{x} \end{aligned}$$

In the last term $\int d\mathbf{x} \propto$ Volume of space: Must be universal (the same for all states).

$$\kappa \equiv \frac{\Lambda^2}{C_F g} \frac{1}{|\mathbf{x}_1 - \mathbf{x}_2|} \quad \text{gives the energy density of the ground state: } \mathcal{E}_\Lambda = \frac{\Lambda^4}{2C_F g^2}$$

$$\text{and potential energy } V(\mathbf{x}_1 - \mathbf{x}_2) = \Lambda^2 |\mathbf{x}_1 - \mathbf{x}_2| - C_F \frac{\alpha_s}{|\mathbf{x}_1 - \mathbf{x}_2|} \quad \Lambda \text{ is the only free parameter.}$$

Quarkonia

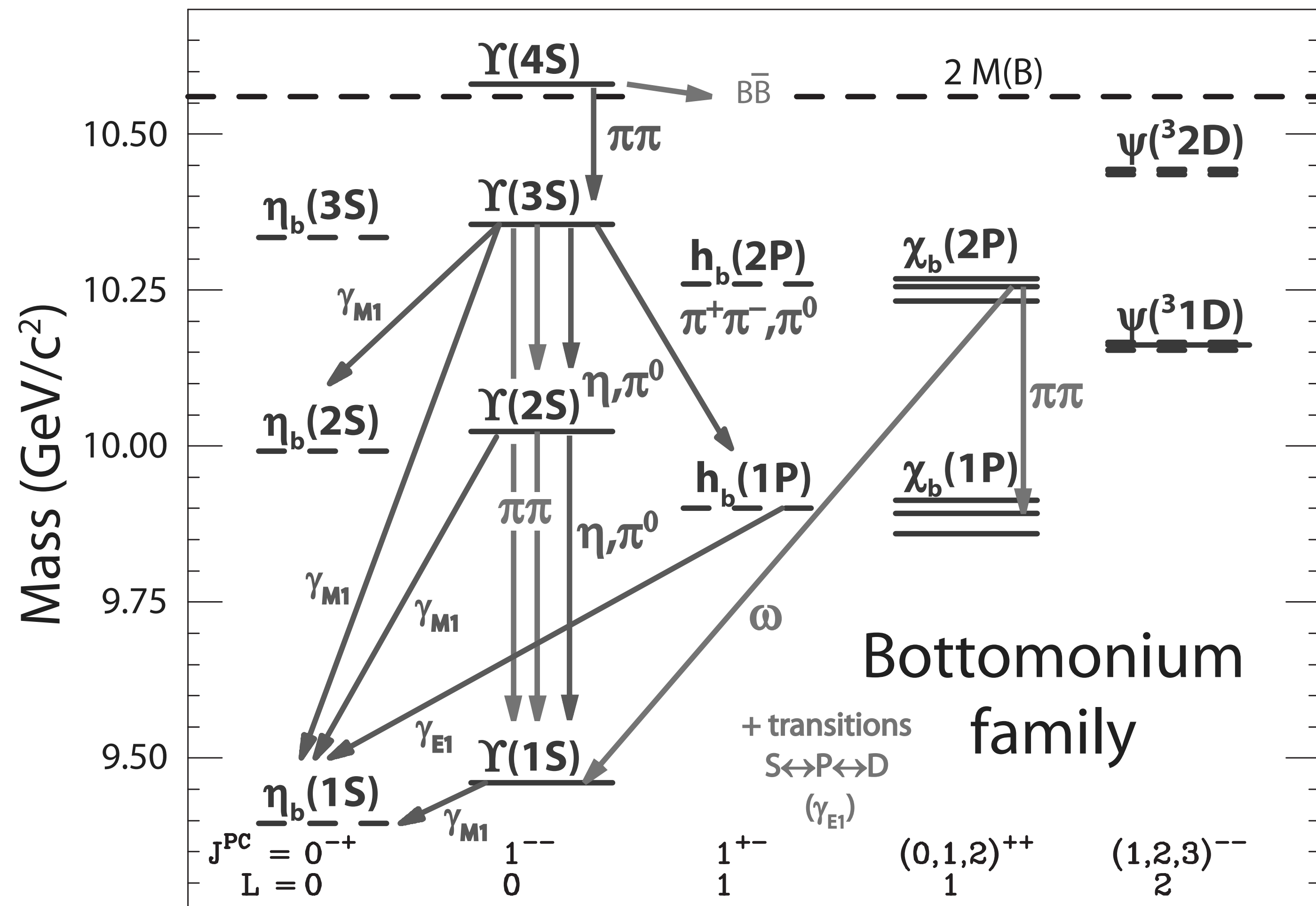
The $c\bar{c}$ and $b\bar{b}$ states are described by the **Schrödinger eq.** with the “Cornell potential”

$$V(r) = V' r - \frac{4}{3} \frac{\alpha_s}{r} \quad (1980)$$

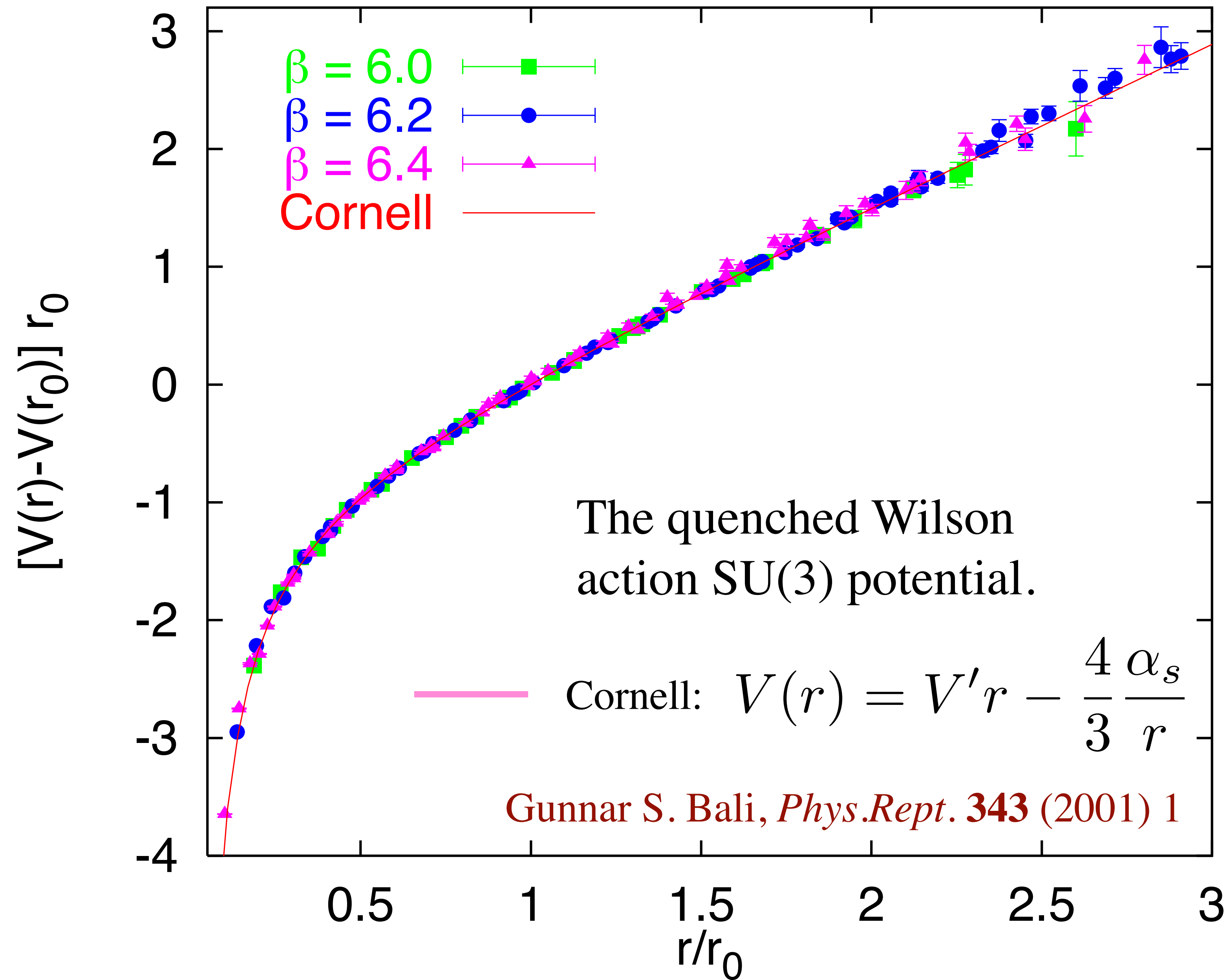
$$V' \simeq 0.18 \text{ GeV}^2, \quad \alpha_s(m_c^2) \simeq 0.39$$

$V' r$ is treated as a **classical potential**.
It involves the **confinement scale V'** .

Transitions are calculated
perturbatively in α_s , as in QED.



Lattice QCD confirmed the Cornell potential



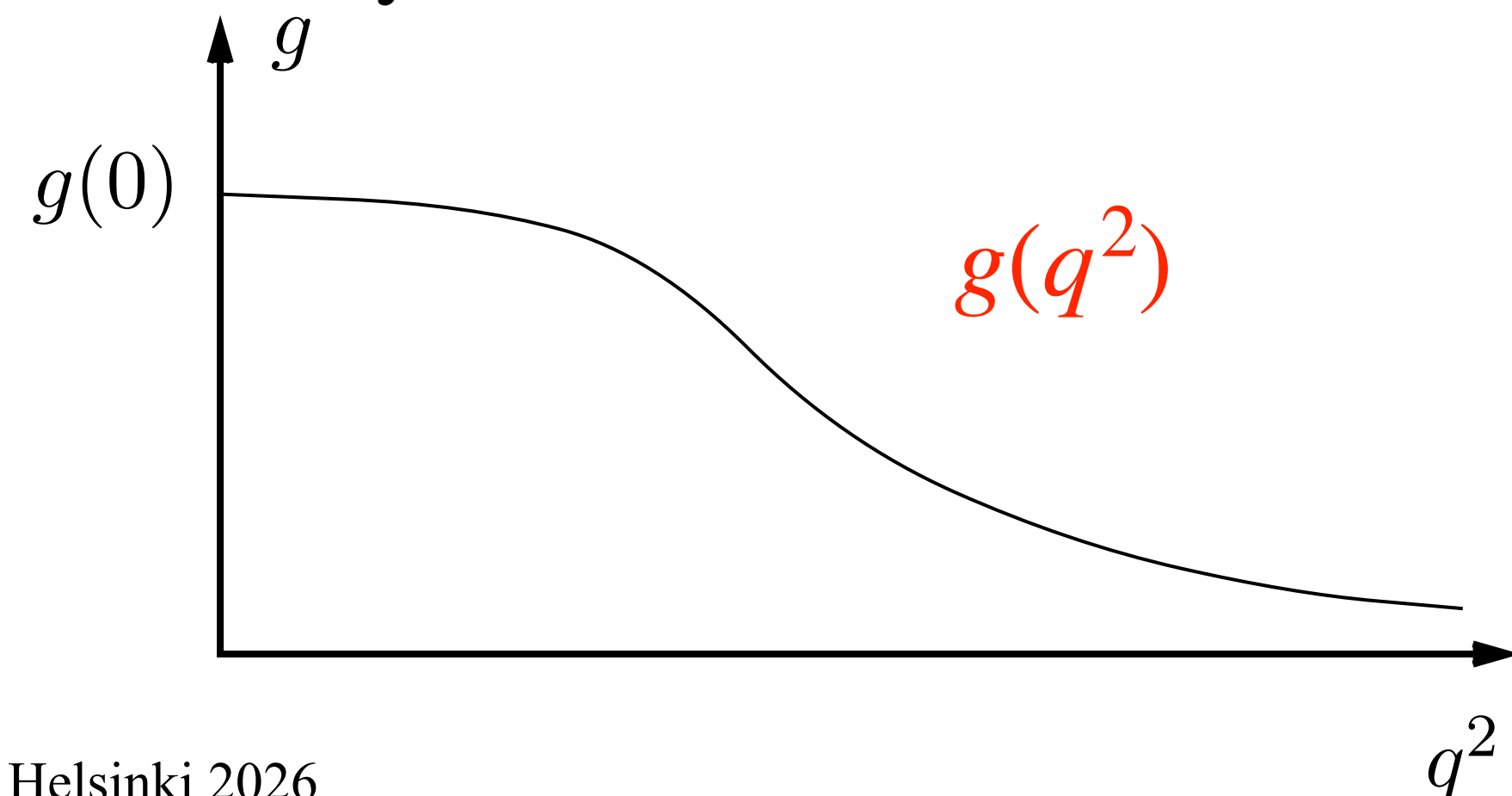
Gribov: Freezing of α_s

There is a critical value of α_s :

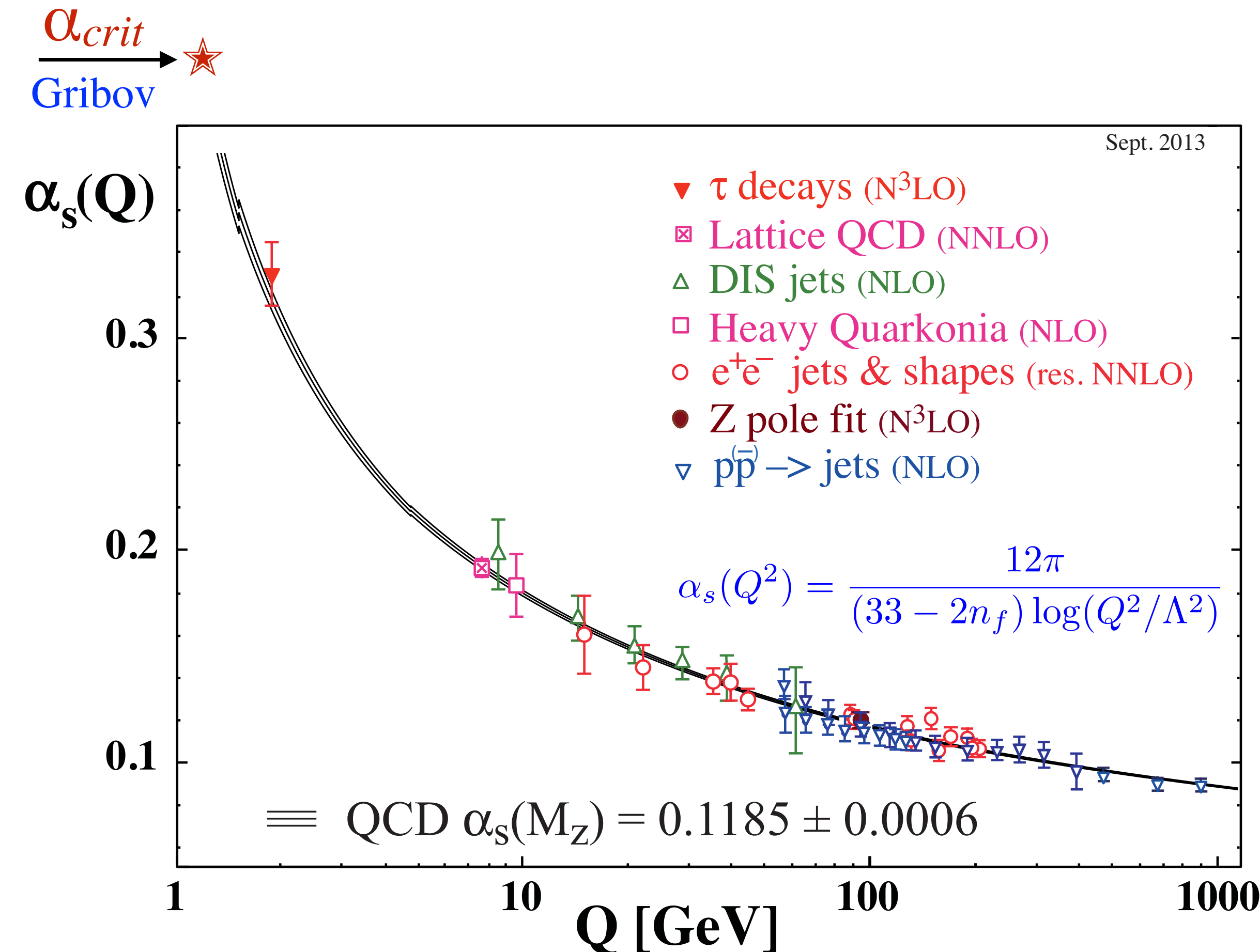
$$\alpha_s^{crit} = \pi C_F^{-1} (1 - \sqrt{2/3}) \simeq 0.43$$

At α_s^{crit} the structure of the vacuum changes and the coupling freezes.

D&K: “This number ... is numerically small. Gribov’s ideas ... offer an intriguing possibility to address all the diversity and complexity of the hadron world from within the field theory with a reasonably small effective interaction strength.”



Gribov, hep-ph/9807224 and hep-ph/9902279
Dokshitzer and Kharzeev, hep-ph/0404216



$$q(\mathbf{x}_1)q(\mathbf{x}_2)q(\mathbf{x}_3): \quad V_{qqq} = \frac{\Lambda^2}{\sqrt{2}} \sqrt{(\mathbf{x}_1 - \mathbf{x}_2)^2 + (\mathbf{x}_2 - \mathbf{x}_3)^2 + (\mathbf{x}_3 - \mathbf{x}_1)^2} - \frac{2}{3} \alpha_s \left(\frac{1}{|\mathbf{x}_1 - \mathbf{x}_2|} + \frac{1}{|\mathbf{x}_2 - \mathbf{x}_3|} + \frac{1}{|\mathbf{x}_3 - \mathbf{x}_1|} \right)$$

$$g(\mathbf{x}_1)g(\mathbf{x}_2): \quad V_{gg} = \sqrt{\frac{N_c}{C_F}} \Lambda^2 |\mathbf{x}_1 - \mathbf{x}_2| - N_c \frac{\alpha_s}{|\mathbf{x}_1 - \mathbf{x}_2|}$$

$$q(\mathbf{x}_1)\bar{q}(\mathbf{x}_2)g(\mathbf{x}_g): \quad V_{q\bar{q}g} = \frac{\Lambda^2}{\sqrt{C_F}} \sqrt{\frac{1}{4} \left(N_c - \frac{2}{N_c} \right) (\mathbf{x}_1 - \mathbf{x}_2)^2 + N_c \left(\mathbf{x}_g - \frac{1}{2} \mathbf{x}_1 - \frac{1}{2} \mathbf{x}_2 \right)^2} + \frac{\alpha_s}{2} \left[\frac{1}{N_c} \frac{1}{|\mathbf{x}_1 - \mathbf{x}_2|} - N_c \left(\frac{1}{|\mathbf{x}_1 - \mathbf{x}_g|} + \frac{1}{|\mathbf{x}_2 - \mathbf{x}_g|} \right) \right]$$

$$V_{qqq}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_2) = V_{q\bar{q}}(\mathbf{x}_1, \mathbf{x}_2)$$

Consistency conditions:

$$V_{q\bar{q}g}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_2) = V_{q\bar{q}}(\mathbf{x}_1, \mathbf{x}_2)$$

$$V_{q\bar{q}g}(\mathbf{x}_1, \mathbf{x}_1, \mathbf{x}_g) = V_{gg}(\mathbf{x}_1, \mathbf{x}_g)$$

Bound state equation for mesons

$$|q\bar{q}; M, \mathbf{P}\rangle \equiv \int d\mathbf{x}_1 d\mathbf{x}_2 \bar{\psi}(\mathbf{x}_1) e^{i\mathbf{P} \cdot (\mathbf{x}_1 + \mathbf{x}_2)/2} \Phi^{(P)}(\mathbf{x}_1 - \mathbf{x}_2) W_\kappa[\mathbf{A}_L; \mathbf{x}_1, \mathbf{x}_2] \psi(\mathbf{x}_2) |0\rangle$$

BSE: $H_{QCD} |q\bar{q}; M, \mathbf{P}\rangle = E_P |q\bar{q}; M, \mathbf{P}\rangle$

$|q\bar{q}\rangle$
 A_L
 $\mathcal{O}(\alpha_s^0)$

$|q\bar{q}g\rangle$
 A_T A_L
 $\mathcal{O}(\alpha_s)$

+ ...

$\mathcal{O}(\alpha_s^0)$: $V(\mathbf{x}) = \Lambda^2 |\mathbf{x}|$, with $g = 0$ in H_{QCD} Lowest order in perturbative expansion

$$i\nabla \cdot \{\boldsymbol{\alpha}, \Phi^{(P)}(\mathbf{x})\} - \frac{1}{2} \mathbf{P} \cdot [\boldsymbol{\alpha}, \Phi^{(P)}(\mathbf{x})] + m[\gamma^0, \Phi^{(P)}(\mathbf{x})] = [E_P - V(\mathbf{x})] \Phi^{(P)}(\mathbf{x}) \quad \text{Poincaré covariant}$$

$\mathbf{P} = 0$: Radial & angular variables separate, J^{PC} of states determined

No quark model exotic states

Chiral symmetry for $m = 0$: Parity doublets

Example: $-\eta_P = \eta_C = (-1)^j$ states at $\mathcal{O}(\alpha_s^0)$, $P = 0$

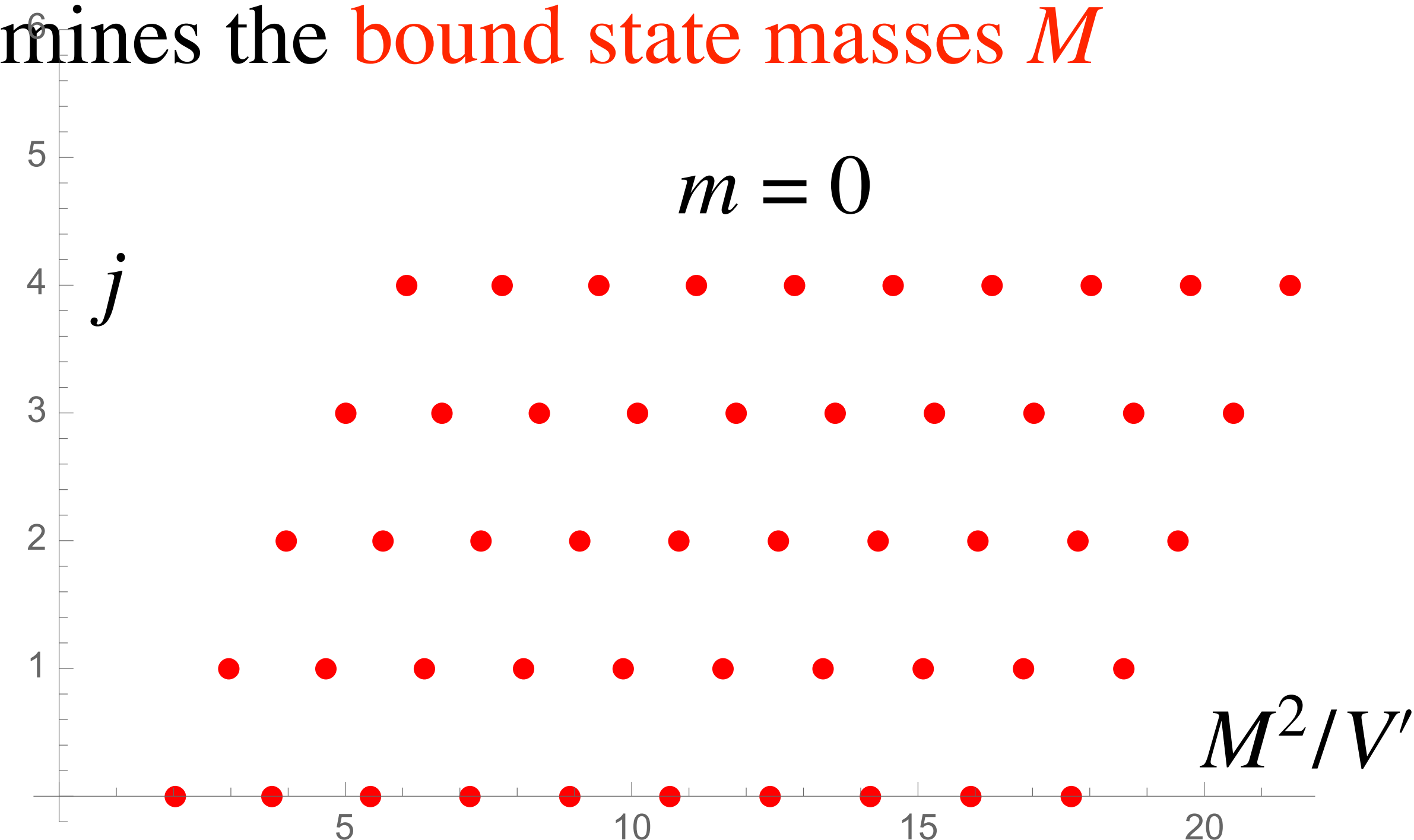
$$\Phi_{-+}(\boldsymbol{x}) = \left[\frac{2}{M - V} (i\boldsymbol{\alpha} \cdot \vec{\nabla} + m\gamma^0) + 1 \right] \gamma_5 F_1(r) Y_{j\lambda}(\hat{\boldsymbol{x}})$$

$$F_1'' + \left(\frac{2}{r} + \frac{V'}{M - V} \right) F_1' + \left[\frac{1}{4}(M - V)^2 - m^2 - \frac{j(j+1)}{r^2} \right] F_1 = 0 \quad \text{Radial equation}$$

Regularity of the wave function determines the **bound state masses M**

Linear Regge trajectories
with daughters:

**Spectrum similar to
dual models**



Breaking of chiral symmetry at $m_q = 0$

Massless ($M = 0$) $J^{PC} = 0^{++}$ state σ for $m = \mathbf{P} = 0$: Poincaré invariant, can mix with vacuum

$$\hat{\sigma} |0\rangle = \int d\mathbf{x}_1 d\mathbf{x}_2 \bar{q}(\mathbf{x}_1) \Phi_\sigma(\mathbf{x}_1 - \mathbf{x}_2) W_\kappa[A_L; \mathbf{x}_1, \mathbf{x}_2] q(\mathbf{x}_2) |0\rangle \quad q = \begin{pmatrix} u \\ d \end{pmatrix} \text{ massless isospin doublet}$$

$$\text{BSE: } i\nabla \cdot \{\boldsymbol{\alpha}, \Phi_\sigma(\mathbf{x})\} + V' r \Phi_\sigma(\mathbf{x}) = 0 \quad \Phi_\sigma(\mathbf{x}) = N_\sigma \left[J_0\left(\frac{1}{4} V' r^2\right) + \frac{i\boldsymbol{\alpha} \cdot \mathbf{x}}{r} J_1\left(\frac{1}{4} V' r^2\right) \right]$$

$$\text{Generator of chiral transformations: } Q_{5j} = \int d\mathbf{x} q^\dagger(\mathbf{x}) \gamma_5 \frac{1}{2} \tau^j q(\mathbf{x})$$

$$\text{Massless } J^{PC} = 0^{-+} \text{ pion: } i[Q_{5j}, \hat{\sigma}] = \hat{\pi}_j \quad \text{with } \Phi_{\pi j}(\mathbf{x}) = -i\tau^j \gamma_5 \Phi_\sigma(\mathbf{x})$$

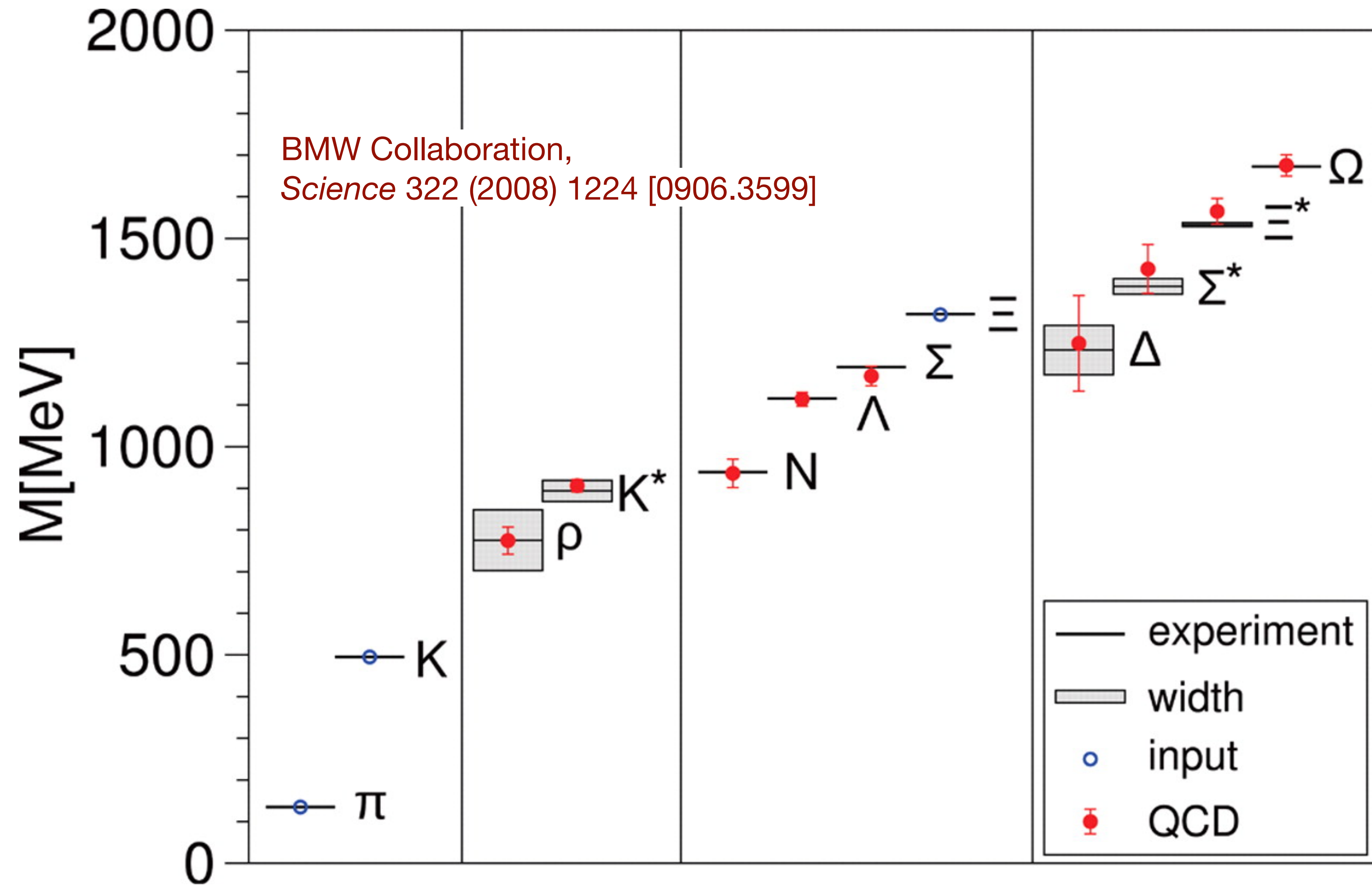
Define σ condensate vacuum: $|\sigma\rangle = \exp(\hat{\sigma}) |0\rangle$ Ground state is not chiral invariant.

Build states with $|0\rangle \rightarrow |\sigma\rangle$: Preserves J^{PC} , but breaks chiral invariance: **no parity doublets.**

Preliminary suggestion for χSB . Requires further study.

Hadron masses from Lattice QCD

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A. S. Kronfeld, *Annu. Rev. Nucl. Part. Sci.* **62** (2012) 265 [1203.1204]