Deformations in AdS/CFT
Integrable spin chains with $U(1)^3$ symmetry

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Plan

- Introduction
- β-deformation and generalisations in gauge theory
- Corresponding deformations in string theory
- Integrability, factorized scattering and the coordinate Bethe ansatz
- Yang-Baxter equation and results for integrability
- Conclusion and Outlook

Based on work with C. Kristjansen and T. Månsson [hep-th/0510221]

[Staudacher hep-th/0412188] [Berenstein, Cherkis hep-th/0405215] [Lunin, Maldacena hep-th/0502086] [Frolov hep-th/0503201] [Frolov, Tseytlin, Roiban hep-th/0503192,0507021] [Beisert, Staudacher hep-th/0504190] [Beisert, Roiban hep-th/0505187]

Introduction

Using integrability to study the AdS/CFT duality has been a very succesful approach

- Scaling dimension of long operators found by diagonalising the Dilatation operator using the Bethe ansatz.
- Agrees with the energy of semiclassical spinning strings up to 3 loops.
- Agreement on the level of actions, etc.
- Succes largely due to integrability

Introduction

Gauge-string duality for less supersymmetry?

Marginal deformations of $\mathcal{N} = 4$ with deformation parameter β, also called β-deformations [Leigh, Strassler]

$\Leftrightarrow$

Strings in the Lunin-Maldacena background $AdS_5 \times S^5_\beta$ [Lunin, Maldacena]

Possible to define semiclassical strings on this background, string energies typically of the form

$$E = J\left(1 + \lambda'(e_1 + e_2(\beta J) + e_3(\beta J)^2) + \mathcal{O}(\lambda'^2)\right) \qquad \lambda' = \frac{\lambda}{J^2}$$

Also: Extension to three deformation parameters β_1, β_2 and β_3. [Frolov] [Beisert, Roiban]

Parameters are allowed to be complex.

The Lunin-Maldacena background

Obtained by deforming the string sigma model in $AdS_5 \times S^5$ by making a TsT transformation.
Sigma model on S^5:

$$S_{S^5} = -\frac{\sqrt{\lambda}}{2}\int d\tau \int \frac{d\sigma}{2\pi}\Big(\gamma^{\alpha\beta}\partial_\alpha r_i \partial_\beta r_i + r_i^2 \partial_\alpha \phi_i \partial_\beta \phi_i + \Lambda(r_i^2 - 1)\Big)$$

Original proposal: Change of variables $\phi_1 = \varphi_3 - \varphi_2$, $\phi_2 = \varphi_1 + \varphi_2 + \varphi_3$, $\phi_3 = \varphi_3 - \varphi_1$

1) T-duality on circle parametrised by φ_1

2) Shift $\varphi_2 \to \varphi_2 + \hat{\gamma}\varphi_1$

3) T-duality on circle parametrised by φ_1

$$S = -\frac{\sqrt{\lambda}}{2}\int d\tau \int \frac{d\sigma}{2\pi}\gamma^{\alpha\beta}\Big[\Big(\partial_\alpha r_i \partial_\beta r_i + G r_i^2 \partial_\alpha \phi_i \partial_\beta \phi_i + \hat{\gamma}^2 G r_1^2 r_2^2 r_3^2 \sum_i \partial_\alpha \phi_i \sum_j \partial_\beta \phi_j\Big)$$

$$-2\hat{\gamma} G \epsilon^{\alpha\beta}(r_1^2 r_2^2 \partial_\alpha \phi_1 \partial_\beta \phi_2 + r_2^2 r_3^2 \partial_\alpha \phi_2 \partial_\beta \phi_3 + r_3^2 r_1^2 \partial_\alpha \phi_3 \partial_\beta \phi_1) + \Lambda(r_i^2 - 1)\Big]$$

$$G^{-1} = 1 + \hat{\gamma}^2(r_1^2 r_2^2 + r_1^2 r_3^2 + r_2^2 r_3^2)$$

The Lunin-Maldacena background

This can be generalized to generate a three parameter deformation. Apply a sequence of TsT dualities:

- TsT on (ϕ_1, ϕ_2) T-duality on ϕ_1 and shift by $\hat{\gamma}_3$ on ϕ_2
- TsT on (ϕ_2, ϕ_3) T-duality on ϕ_2 and shift by $\hat{\gamma}_1$ on ϕ_3
- TsT on (ϕ_3, ϕ_1) T-duality on ϕ_3 and shift by $\hat{\gamma}_2$ on ϕ_1

The dual background for complex parameters, $\beta_i = \hat{\gamma}_i + i\hat{\sigma}_i$, is found by performing $SL(2,\mathbb{R})$ transformations. I.e. a sequence of $S_\sigma T s_\gamma T S_\sigma^{-1}$ gives the three complex parameter background

β-deformed $\mathcal{N}=4$ SYM

Superpotential in $\mathcal{N}=4$

$$W_{\mathcal{N}=4} = \text{Tr}(\Phi_1\Phi_2\Phi_3 - \Phi_1\Phi_3\Phi_3)$$

Two exactly marginal deformations in $\mathcal{N}=4$

$$W_{def} = \text{Tr}(e^{i\pi\beta}\Phi_1\Phi_2\Phi_3 - e^{-i\pi\beta}\Phi_1\Phi_3\Phi_3) + h'\text{Tr}(\Phi_1^3 + \Phi_2^3 + \Phi_3^3)$$

The resulting theory is $\mathcal{N}=1$ supersymmetric and conformal.

Set $h'=0$.

β-deformed $\mathcal{N} = 4$ SYM

In terms of component fields

$$V = \mathrm{Tr}\Big(|e^{i\pi\beta}\Phi_1\Phi_2 - e^{-i\pi\beta}\Phi_2\Phi_1|^2 + |e^{i\pi\beta}\Phi_2\Phi_3 - e^{-i\pi\beta}\Phi_3\Phi_2|^2$$

$$+|e^{i\pi\beta}\Phi_3\Phi_1 - e^{-i\pi\beta}\Phi_1\Phi_3|^2\Big) + \mathrm{Tr}\left([\Phi_1,\bar{\Phi}_1]^2 + [\Phi_2,\bar{\Phi}_2]^2 + [\Phi_3,\bar{\Phi}_3]^2\right)$$

Introduce a more general deformation 3 (6) parameter deformation

$$V = \mathrm{Tr}\Big(|e_1^{i\pi\beta_1}\Phi_1\Phi_2 - e^{-i\pi\beta_1}\Phi_2\Phi_1|^2 + |e^{i\pi\beta_2}\Phi_2\Phi_3 - e^{-i\pi\beta_2}\Phi_3\Phi_2|^2$$

$$+|e^{i\pi\beta_3}\Phi_3\Phi_1 - e^{-i\pi\beta_3}\Phi_1\Phi_3|^2\Big) + \mathrm{Tr}\left([\Phi_1,\bar{\Phi}_1]^2 + [\Phi_2,\bar{\Phi}_2]^2 + [\Phi_3,\bar{\Phi}_3]^2\right)$$

$\beta_i \in \mathbb{C}$ This deformation is not supersymmetric but conformal.

Dilatation operator in the deformed theory

Consider operators in $\mathcal{N} = 4$ of the form

$$\mathcal{O}(x) = \text{Tr}(X^{J_1} Y^{J_2} Z^{J_3} + \ldots) \quad X,\, Y,\, Z \text{ chiral scalars}$$

Dilatation operator associated with $\mathfrak{su}(3)$ nearest neighbour ferromagnetic spin chain.

$$\mathcal{D} = \frac{\lambda}{8\pi^2} \sum_{k=1}^{J} H_{k,k+1} = \frac{\lambda}{8\pi^2} \sum_{k=1}^{J} (1_{k,k+1} - P_{k,k+1})$$

[Minahan, Zarembo].

This is generalized to the full theory giving the dilatation operator in $\mathfrak{psu}(2,2|4)$.
Higher loops introduce interactions beyond nearest neighbours.
We can write the $\mathfrak{su}(3)$ hamiltonian in terms of the generators

$$E_{ij}|k\rangle = \delta_{jk}|i\rangle$$

Dilatation operator

The $\mathfrak{su}(3)$ sector in $\mathcal{N}=4$

$$H^{su(3)}_{k,k+1} = E^k_{00}E^{k+1}_{11} + E^k_{11}E^{k+1}_{00} + E^k_{00}E^{k+1}_{22} + E^k_{22}E^{k+1}_{00} + E^k_{11}E^{k+1}_{22} + E^k_{22}E^{k+1}_{11}$$
$$-E^k_{12}E^{k+1}_{21} - E^k_{21}E^{k+1}_{12} - E^k_{10}E^{k+1}_{01} - E^k_{01}E^{k+1}_{10} - E^k_{20}E^{k+1}_{02} - E^k_{02}E^{k+1}_{20}$$

On matrix form

$$H^{su(3)} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

The deformed dilatation operator

Use the notation $q_i = e^{i\pi\beta_i} = r_i e^{i\gamma_i}$.

$$H^{su(3)}_{k,k+1} = E^k_{00}E^{k+1}_{11} + r_2^2 E^k_{11}E^{k+1}_{00} + r_3^2 E^k_{00}E^{k+1}_{22} + E^k_{22}E^{k+1}_{00} + E^k_{11}E^{k+1}_{22} + r_1^2 E^k_{22}E^{k+1}_{11}$$
$$-r_1 e^{-i\gamma_1} E^k_{12}E^{k+1}_{21} - r_1 e^{i\gamma_1} E^k_{21}E^{k+1}_{12} - r_2 e^{i\gamma_2} E^k_{10}E^{k+1}_{01} - r_2 e^{-i\gamma_2} E^k_{01}E^{k+1}_{10}$$
$$-r_3 e^{-i\gamma_3} E^k_{20}E^{k+1}_{02} - r_3 e^{i\gamma_3} E^k_{02}E^{k+1}_{20}$$

$$H^{su(3)} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & r_3 e^{-i\gamma_3} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & r_3^2 & 0 & 0 & 0 & r_2 e^{i\gamma_2} & 0 & 0 \\ 0 & r_3 e^{i\gamma_3} & 0 & r_2^2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & r_1 e^{-i\gamma_1} & 0 \\ 0 & 0 & r_2 e^{-i\gamma_2} & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & r_1 e^{i\gamma_1} & 0 & r_1^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Integrability?

Is the model integrable?

- $\mathfrak{su}(2)$: Yes, always! [Berenstein, Cherkis]
- $\mathfrak{su}(3)$: Yes, when $r_i = 1$ [Beisert, Roiban]
  No, when $r_1 = r_2 = r_3 = r$, $\gamma_1 = \gamma_2 = \gamma_3 = \gamma$ [Berenstein, Cherkis]
  Maybe not when $r_i \neq 1$, $\gamma_1 \neq \gamma_2 \neq \gamma_3 \ldots$

Investigate this!

More general: Any Hamiltonian with $U(1)^3$ symmetry

$$
\begin{aligned}
H^{k,k+1} &= H^{00}_{00} E^k_{00} E^{k+1}_{00} + H^{11}_{11} E^k_{11} E^{k+1}_{11} + H^{22}_{22} E^k_{22} E^{k+1}_{22} + H^{12}_{12} E^k_{11} E^{k+1}_{22} + H^{21}_{12} E^k_{12} E^{k+1}_{21} \\
&+ H^{12}_{21} E^k_{21} E^{k+1}_{12} + H^{21}_{21} E^k_{22} E^{k+1}_{11} + H^{01}_{10} E^k_{10} E^{k+1}_{01} + H^{10}_{10} E^k_{11} E^{k+1}_{00} + H^{01}_{01} E^k_{00} E^{k+1}_{11} \\
&+ H^{10}_{01} E^k_{01} E^{k+1}_{10} + H^{02}_{20} E^k_{20} E^{k+1}_{02} + H^{20}_{20} E_{22} E_{00} + H^{02}_{02} E_{00} E_{22} + H^{20}_{02} E_{02} E_{20} ,
\end{aligned}
$$

A general Hamiltonian

$$H = \begin{pmatrix} H^{00}_{00} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & H^{01}_{01} & 0 & H^{01}_{10} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & H^{02}_{02} & 0 & 0 & 0 & H^{02}_{20} & 0 & 0 \\ 0 & H^{10}_{01} & 0 & H^{10}_{10} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & H^{11}_{11} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & H^{12}_{12} & 0 & H^{12}_{21} & 0 \\ 0 & 0 & H^{20}_{02} & 0 & 0 & 0 & H^{20}_{20} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & H^{21}_{12} & 0 & H^{21}_{21} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & H^{22}_{22} \end{pmatrix}.$$

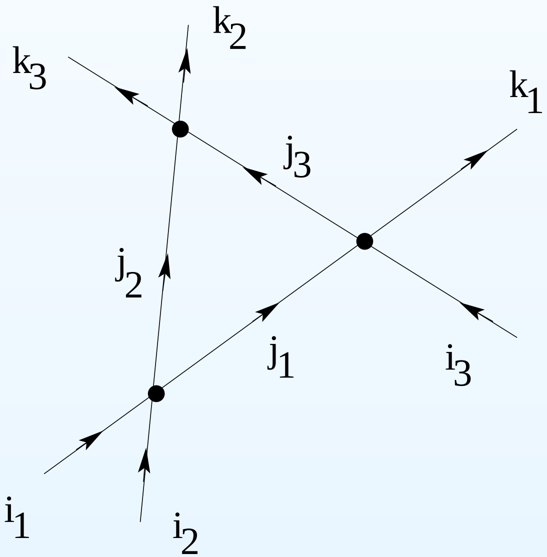
Require hermiticity: $H^{21}_{12} = (H^{12}_{21})^* = r_1 e^{i\gamma_1}$, $H^{02}_{20} = (H^{20}_{02})^* = r_2 e^{i\gamma_2}$, $H^{10}_{01} = (H^{01}_{10})^* = r_3 e^{i\gamma_3}$, diagonal terms real. Not all parameters are physical, we are allowed to rescale and add/subtract number operators.$\Rightarrow$ 9 physical parameters

When is *this* model integrable?

Investigating integrability

Integrability $\Leftrightarrow$ Factorized scattering

Consider an N particle process: scattering occurs as a sequence of two-particle scatterings, in the case of 3 particles:



Alternative more technical definition: Existence of an R-matrix that satisfies the Yang-Baxter equation

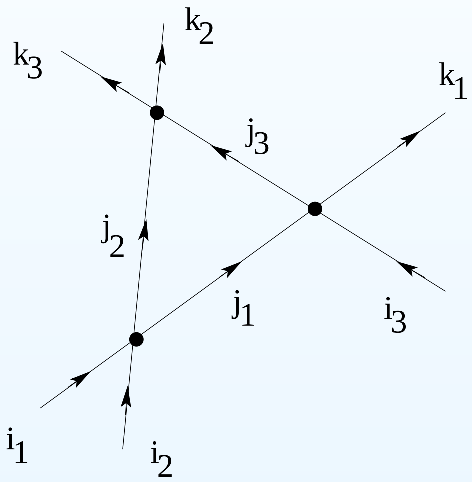
$$R^{12}R^{13}R^{23} = R^{23}R^{13}R^{12}$$

leads to an infinite number of commuting charges.

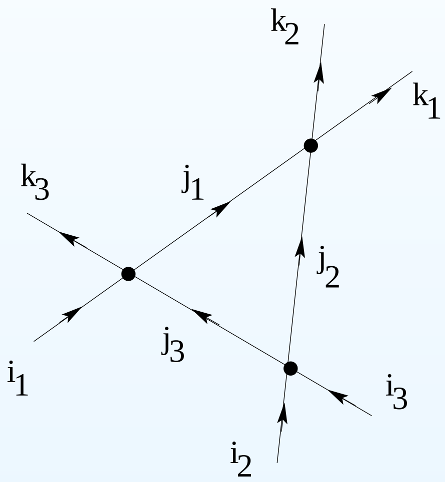
Investigating integrability

Yang-Baxter equation represented graphically

$$\sum_{j_1,j_2,j_3}$$



$$= \sum_{j_1,j_2,j_3}$$



$$R^{j_1 j_2}_{i_1 i_2}(u-v) R^{k_1 j_3}_{j_1 i_3}(u) R^{k_2 k_3}_{j_2 j_3}(v) = R^{j_2 j_3}_{i_2 i_3}(v) R^{j_1 k_3}_{i_1 j_3}(u) R^{k_1 k_2}_{j_1 j_2}(u-v)$$

Factorized scattering leads to a similar relation for the S-matrix

$$S^{j_1 j_2}_{i_1 i_2}(p_{i_1}, p_{i_2}) S^{k_1 j_3}_{j_1 i_3}(p_{i_1}, p_{i_3}) S^{k_2 k_3}_{j_2 j_3}(p_{i_2}, p_{i_3}) = S^{j_2 j_3}_{i_2 i_3}(p_{i_1}, p_{i_2}) S^{j_1 k_3}_{i_1 j_3}(p_{i_1}, p_{i_3}) S^{k_1 k_2}_{j_1 j_2}(p_{i_2}, p_{i_3})$$

Eliminating the phases in the Hamiltonian

$$\tilde{H}_{ij}^{kl} = \exp\left(\frac{i}{2}(\epsilon_{ijm}\gamma^m - \epsilon_{kln}\gamma^n)\right) H_{ij}^{kl}$$

Corresponding R-matrices

$$\tilde{R}_{ij}^{kl} = \exp\left(\frac{i}{2}(\epsilon_{ijm}\gamma^m - \epsilon_{kln}\gamma^n)\right) R_{ij}^{lk}$$

If the Hamiltonian is integrable without phases it is also integrable with.

$$\tilde{\tilde{H}}{}_{ij}^{kl} = \exp\left(-\frac{i}{2}(\epsilon_{ijm}\gamma^m - \epsilon_{kln}\gamma^n)\right) H_{ij}^{kl}$$

Corresponding R-matrices

$$\tilde{\tilde{R}}{}_{ij}^{kl} = \exp\left(-\frac{i}{2}(\epsilon_{ijm}\gamma^m - \epsilon_{kln}\gamma^n)\right) R_{ij}^{lk}$$

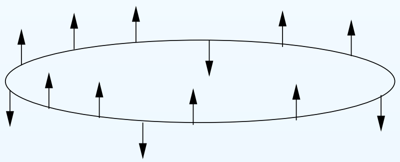
If the Hamiltonian is integrable with phases it is also integrable without.

The S-matrix

Find the S-matrix and investigate when it satisfies the Yang-Baxter equation!

What do we scatter? Excitations on a spin chain! For our hamiltonian we have two types of excitations (for the $\mathfrak{su}(2)$ sector just one).

$$\mathcal{O}(x) = XXXYXXYX = |\uparrow\uparrow\uparrow\downarrow\uparrow\uparrow\downarrow\uparrow\rangle$$



Tr $\longrightarrow$ periodic chain.

We have

$$\mathcal{O}(x) = XXYZXYXX = |00120100\rangle$$

Spin chain with two types of excitations.

Obtaining the S-matrix

Choose a reference state, $|0000000\ldots\rangle$ for instance. One-excitation states

$$|00000{\color{red}1}00000\rangle \quad |00000{\color{red}2}00000\rangle.$$

Consider scattering of excitations:

$$|000000{\color{red}1}0000{\color{red}1}00000\rangle$$

$$|000000{\color{red}2}0000{\color{red}2}00000\rangle$$

$$|000000{\color{red}1}0000{\color{red}2}00000\rangle$$

$$|000000{\color{red}2}0000{\color{red}1}00000\rangle$$

Act with the Hamiltonian on eigenstates

$$|ij\rangle = \sum_{1\leq l_1<l_2\leq L} \psi_{ij}(l_1,l_2)|00 \overset{\overset{l_1}{\downarrow}}{i}\ 000\ \overset{\overset{l_2}{\downarrow}}{j}\ 000\ldots\rangle$$

Infinitely long chains. Need an ansatz $\psi_{ij}(l_1,l_2)$.

Obtaining the S-matrix

Case of two particles of the same type, particles exchange momenta as they scatter

$$\psi_{11}(l_1,l_2) = e^{ip_1 l_1 + ip_2 l_2} + d(p_2,p_1) e^{ip_2 l_1 + ip_1 l_2}$$

Apply H to this

$$H = H^{00}_{00} \underbrace{E^k_{00} E^{k+1}_{00}}_{00\to 00} + H^{11}_{11} \underbrace{E^k_{11} E^{k+1}_{11}}_{11\to 11} + H^{10}_{10} \underbrace{E^k_{11} E^{k+1}_{00}}_{10\to 01} + H^{01}_{01} \underbrace{E^k_{00} E^{k+1}_{11}}_{01\to 01} + r_3 \underbrace{E^k_{10} E^{k+1}_{01}}_{01\to 10} + r_3 \underbrace{E^k_{01} E^{k+1}_{10}}_{10\to 01} + \ldots$$

$l_2 > l_1 + 1:$

$$E_{11}\psi_{11}(l_1,l_2) = \left(H^{00}_{00}(L-4) + 2H^{10}_{10} + 2H^{01}_{01}\right)\psi_{11}(l_1,l_2) + r_3\{\psi_{11}(l_1+1,l_2) + \psi_{11}(l_1,l_2+1)$$
$$+\psi_{11}(l_1-1,l_2) + \psi_{11}(l_1,l_2-1)\}$$

$l_2 = l_1 + 1:$

$$E_{11}\psi_{11}(l_1,l_2) = \left(H^{00}_{00} + H^{11}_{11} + H^{10}_{10} + H^{01}_{01}\right)\psi_{11}(l_1,l_2) + r_3\{\psi_{11}(l_1,l_2+1) + \psi_{11}(l_1-1,l_2)\}$$

Obtaining the S-matrix

We obtain the energy

$$E_{11} = H^{00}_{00}(L-4) + 2H^{10}_{10} + 2H^{01}_{01} + r_2\left(e^{ip_1} + e^{ip_2} + e^{-ip_1} + e^{-ip_2}\right),$$

...and (part of) the S-matrix

$$d(p_1,p_2) = -\frac{s_1 e^{ip_1} + e^{ip_1+ip_2} + 1}{s_1 e^{ip_2} + e^{ip_1+ip_2} + 1} \quad s_1 = (H^{10}_{10} - H^{00}_{00} - H^{11}_{11} + H^{01}_{01})$$

Same thing for $2-2$ scattering

$$E_{22} = H^{00}_{00}(L-4) + 2H^{20}_{20} + 2H^{02}_{02} + r_2\left(e^{ip_1} + e^{ip_2} + e^{-ip_1} + e^{-ip_2}\right),$$

$$a(p_1,p_2) = -\frac{s_2 e^{ip_1} + e^{ip_1+ip_2} + 1}{s_2 e^{ip_2} + e^{ip_1+ip_2} + 1}, \quad s_2 = (H^{20}_{20} - H^{00}_{00} - H^{22}_{22} + H^{02}_{02})/r_2.$$

Obtaining the S-matrix

Scattering of two different particles

$$H|\psi\rangle = H \begin{pmatrix} |12\rangle \\ |21\rangle \end{pmatrix} = E|\psi\rangle$$

$l_2 > l_1 + 1:$

$$\begin{aligned}
E\psi_{12}(l_1,l_2) &= \left(H^{00}_{00}(L-4) + H^{10}_{10} + H^{01}_{01} + H^{20}_{20} + H^{02}_{02}\right)\psi_{12}(l_1,l_2) \\
&\quad + r_2\left(\psi_{12}(l_1+1,l_2) + \psi_{12}(l_1-1,l_2)\right) \\
&\quad + r_3\left(\psi_{12}(l_1,l_2+1) + \psi_{12}(l_1-1,l_2)\right), \\
E\psi_{21}(l_1,l_2) &= \left(H^{00}_{00}(L-4) + H^{10}_{10} + H^{01}_{01} + H^{20}_{20} + H^{02}_{02}\right)\psi_{21}(l_1,l_2) \\
&\quad + r_2\left(\psi_{21}(l_1,l_2+1) + \psi_{21}(l_1,l_2-1)\right) \\
&\quad + r_3\left(\psi_{21}(l_1+1,l_2) + \psi_{21}(l_1-1,l_2)\right).
\end{aligned}$$

Obtaining the S-matrix

$$l_2 = l_1 + 1:$$

$$\begin{aligned} E\psi_{12}(l_1, l_2) &= \left(H_{00}^{00} + H_{12}^{12} + H_{01}^{01} + H_{20}^{20}\right)\psi_{12}(l_1, l_2) \\ &\quad + r_1\psi_{21}(l_1, l_2) + r_2\psi_{21}(l_1 - 1, l_2) + r_3\psi_{12}(l_1, l_2 + 1), \\ E\psi_{21}(l_1, l_2) &= \left(H_{00}^{00}(L-3) + H_{21}^{21} + H_{10}^{10} + H_{02}^{02}\right)\psi_{21}(l_1, l_2) \\ &\quad + r_1\psi_{12}(l_1, l_2) + r_2\psi_{21}(l_1, l_2 + 1) + r_3\psi_{21}(l_1 - 1, l_2). \end{aligned}$$

Need a more general ansatz here. In general different dispersion relations for different type of particles

$$\psi_{12}(l_1, l_2) = A_{12}e^{ip_1 l_1 + ip_2 l_2} + A'_{12}e^{ip'_1 l_2 + ip'_2 l_1}$$

$$\psi_{21}(l_1, l_2) = A_{21}e^{ip'_1 l_1 + ip'_2 l_2} + A_{21}e^{ip_1 l_2 + ip_2 l_1}$$

Obtaining the S-matrix

We obtain

$$\begin{aligned} E &= H^{00}_{00}(L-4) + H^{10}_{10} + H^{01}_{01} + H^{20}_{20} + H^{02}_{02} + r_2(e^{ip_1} + e^{-ip_1}) + r_3(e^{ip_2} + e^{-ip_2}) \\ &= H^{00}_{00}(L-4) + H^{10}_{10} + H^{01}_{01} + H^{20}_{20} + H^{02}_{02} + r_2(e^{ip'_2} + e^{-ip'_2}) + r_3(e^{ip'_1} + e^{-ip'_1}) \end{aligned}$$

together with the conservation of momenta

$$r_2 \cos p_1 + r_3 \cos p_2 = r_2 \cos p'_2 + r_3 \cos p'_1$$

$$p_1 + p_2 = p'_1 + p'_2.$$

Gives

$$e^{ip'_1} = e^{ip_1} \frac{r_3 + r_2 e^{ip_1 + ip_2}}{r_2 + r_3 e^{ip_1 + ip_2}} \qquad e^{ip'_2} = e^{ip_2} \frac{r_2 + r_3 e^{ip_1 + ip_2}}{r_3 + r_2 e^{ip_1 + ip_2}}$$

Finally the S-matrix

Defined in the transmission diagonal representation

$$\begin{pmatrix} A'_{21} \\ A'_{12} \end{pmatrix} = \begin{pmatrix} c(p_2,p_1) & b(p_2,p_1) \\ \bar{b}(p_2,p_1) & \bar{c}(p_2,p_1) \end{pmatrix} \begin{pmatrix} A_{12} \\ A_{21} \end{pmatrix},$$

$$c(p_1,p_2) = -\frac{1}{D} r_1 (r_3 + r_2 e^{ip_1+ip_2})(e^{ip'_2} - e^{ip_1})$$

$$\bar{c}(p_1,p_2) = -\frac{1}{D} r_1 (r_2 + r_3 e^{ip_1+ip_2})(e^{ip_2} - e^{ip'_1})$$

$$b(p_1,p_2) = \frac{1}{D}\Big(r_1^2 e^{ip_1+ip_2} - (r_1 t_1 e^{ip'_2} + r_2 e^{ip_1+ip_2} + r_3)(r_1 t_2 e^{ip'_1} + r_2 + r_3 e^{ip_1+ip_2})\Big)$$

$$\bar{b}(p_1,p_2) = \frac{1}{D}\left(r_1^2 e^{ip_1+ip_2} - (r_1 t_1 e^{ip_1} + r_2 e^{ip_1+ip_2} + r_3)(r_1 t_2 e^{ip_2} + r_2 + r_3 e^{ip_1+ip_2})\right)$$

where

$$D = (r_1 t_1 e^{ip'_2} + r_2 e^{ip_2+ip_3} + r_3)(r_1 t_2 e^{ip_2} + r_2 + r_3 e^{ip_1+ip_2}) - r_1^2 e^{ip_2+ip'_2}$$

and

$$t_1 = (H^{10}_{10} - H^{00}_{00} - H^{12}_{12} + H^{02}_{02})/r_1$$
$$t_2 = (H^{01}_{01} - H^{00}_{00} - H^{21}_{21} + H^{20}_{20})/r_1.$$

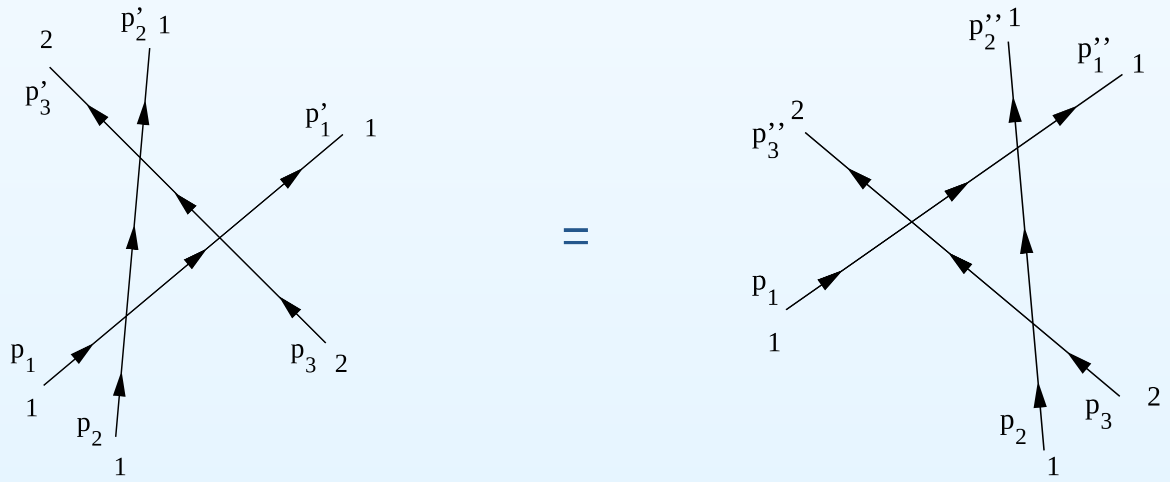
Integrability?

Number of independent parameters in the S-matrix: s_1, s_2, t_1, t_2, r_2/r_1, r_3/r_1, γ_1, γ_2, γ_3 (9)

Integrable with angles $\leftrightarrow$ Integrable without angles

$\longrightarrow$ Explore the remaining $9 - 3 = 6$ dimensional parameter space.

When is the Yang-Baxter equation satisfied?



Quantum number same on in/outgoing states on both sides of the equation.

$\longrightarrow \quad r_1 = r_2 = r_3$

Integrability?

The other cases:

$$S^{12}S^{13}S^{23} = S^{23}S^{13}S^{12}$$

We get 7 independent equations

$$a(p_1,p_2)\bar{b}(p_1,p_3)a(p_2,p_3) - \bar{b}(p_1,p_2)a(p_1,p_3)\bar{b}(p_2,p_3) - c(p_1,p_3)\bar{b}(p_1,p_3)\bar{c}(p_2,p_3) = 0$$

$$\bar{c}(p_1,p_2)a(p_1,p_3)b(p_2,p_3) - a(p_1,p_2)\bar{c}(p_1,p_3)b(p_2,p_3) + \bar{b}(p_1,p_2)b(p_1,p_3)\bar{c}(p_2,p_3) = 0$$

$$\bar{b}(p_1,p_2)b(p_1,p_3)\bar{b}(p_2,p_3) - b(p_1,p_2)\bar{b}(p_1,p_3)b(p_2,p_3) = 0$$

$$\bar{b}(p_1,p_2)c(p_1,p_3)a(p_2,p_3) - c(p_1,p_2)\bar{b}(p_1,p_3)b(p_2,p_3) - \bar{b}(p_1,p_2)a(p_1,p_3)c(p_2,p_3) = 0$$

$$\bar{c}(p_1,p_2)\bar{b}(p_1,p_3)b(p_2,p_3) + \bar{b}(p_1,p_2)d(p_1,p_3)\bar{c}(p_2,p_3) - \bar{b}(p_1,p_2)\bar{c}(p_1,p_3)d(p_2,p_3) = 0$$

$$\bar{b}(p_1,p_2)d(p_1,p_3)\bar{b}(p_2,p_3) + c(p_1,p_2)\bar{b}(p_1,p_3)\bar{c}(p_2,p_3) - d(p_1,p_2)\bar{b}(p_1,p_3)d(p_2,p_3) = 0$$

$$b(p_1,p_2)\bar{b}'c(p_2,p_3) + c(p_1,p_2)d'\bar{b}(p_2,p_3) - d(p_1,p_2)c(p_1,p_3)\bar{b}(p_2,p_3) = 0$$

Integrability?

Families of solutions

1) $r_1 = r_2 = r_3 \quad t_1 t_2 = 1, \quad s_1 = 0, \quad s_2 = 0$

2) $r_1 = r_2 = r_3 \quad t_1 t_2 = 1, \quad s_1 = 0, \quad s_2 = t_1 + 1/t_1$

3) $r_1 = r_2 = r_3 \quad t_1 t_2 = 1, \quad s_1 = t_1 + 1/t_1, \quad s_2 = 0$

4) $r_1 = r_2 = r_3 \quad t_1 t_2 = 1, \quad s_1 = t_1 + 1/t_1, \quad s_2 = t_1 + 1/t_1$

5) $r_1 \neq 0 \quad r_2 = r_3 = 0$

6) $r_1 = 0 \quad r_2 = r_3 = r \neq 0$

Family 4) contains several known cases

- Integrable deformation of $\mathcal{N} = 4$ with phases only
  $r_i = 1, \quad s_1 = s_2 = 2, \quad t_1 = t_2 = 1$ [Beisert, Roiban]
  One parameter deformation [Berenstein, Cherkis]

- $\mathfrak{su}_q(3)$ chain, $r_i = R, \quad s_1 = s_2 = \frac{1+R^2}{2}, \quad t_1 = \frac{2R^2-1}{R}, \quad t_2 = \frac{2-R^2}{2}$

Integrability

Family 3)

- The $\mathfrak{su}(1|2)$ spin chain. $r = 1 \quad s_1 = 2 \quad s_2 = 0 \quad t_1 = t_2 = 1$.
- Several model known from condensed matter theory
- Extension of the integrable case [Beisert, Roiban] with complex phases is not integrable as suspected.

$$s_1 = s_2 = \frac{1 + r^2}{r} \quad t_1 = \frac{2r^2 - 1}{r} \quad t_2 = \frac{2 - r^2}{r}$$

- This model with certain added terms on the diagonal is integrable.

R-matrices

It is possible to use the information about the S-matrices computed with respect to the different reference states to construct an R-matrix. This can be done in all the integrable classes here.

Conclusions and Outlook

- Investigated the gauge theories corresponding to strings in the Lunin-Maldacena background.
- Integrability found only for real deformations in these model. The complex case is not integrable.
- Integrability in classes of models with $U(1)^3$ symmetry.

- Importance for string theory?
- What can be done without integrability?
- Other deformations...
- Non-nearest neighbour interactions? Does integrability remain?