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Dario Martelli
CERN

Recent results in AdS/CFT

DISCLAIMER:
I can't cover all the developments $\Rightarrow$ selected topics.

Plan of the talk

- Introduction

- The example of the conifold

Geometry

- Some Sasaki–Einstein geometry

- $Y^{p,q}$ metrics

- Calabi–Yau cones

- Toric manifolds and singularities

- $L^{a,b,c}$ metrics and toric singularities

Field theory

- $\mathcal{N} = 1$ Quiver gauge theories
- $Y^{p,q}$ quivers
- Baryonic operators and R-charges
- $L^{a,b,c}$ quivers. Basics of "dimers".
- a-maximisation in $\mathcal{N} = 1$ SCFT

The geometric dual of a-maximisation

- Z-minimisation
- relation to a-maximisation

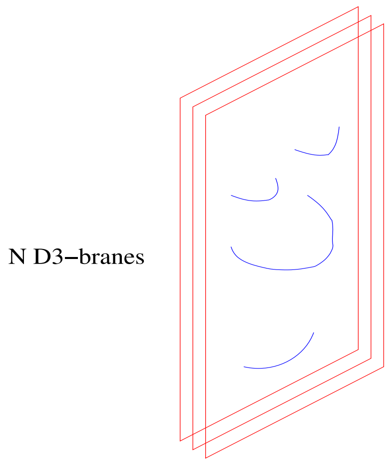
Conclusions

- summary
- outlook and open problems

AdS/CFT correspondence

D-branes in String Theory allow us to construct and study supersymmetric gauge theories.

N parallel D3-branes in $\mathbb{R}^{1,9}$ space-time:



World-volume gauge theory: $\mathcal{N}=4$ $SU(N)$ super–Yang–Mills in 4d.

This theory is conformal: $\beta \equiv 0$

Supergravity description: a large number N of D3-branes back-react on the geometry, curving space-time.

$$ds^2 = H^{-1/2}(r)ds^2(\mathbb{R}^{1,3}) + H^{1/2}(r)(dr^2 + r^2 ds^2(S^5))$$

$$H(r) = 1 + \frac{g_s N(\alpha')^2}{r^4}$$

In the "near-horizon" the geometry becomes $\mathrm{AdS}_5 \times S^5$.

This is a solution of Type IIB supergravity, preserving 32 supersymmetries:

$$ds^2 = ds^2(AdS_5) + ds^2(S^5)$$

$$F^5_{RR} = N(\mathrm{vol}(AdS_5) + \mathrm{vol}(S^5))$$

Maldacena's conjecture: the large N limit (and $g_s N >> 1$) of $\mathcal{N} = 4$ SYM is dual to Type IIB string theory in the $\mathrm{AdS}_5 \times S^5$ background geometry.

Symmetry of the geometry: $SO(2,4) \times SO(6)$

$SO(2,4)$ is the conformal group in 4 dimensions

$SO(6) \simeq SU(4)$ is an internal symmetry of $\mathcal{N} = 4$ SYM theory that rotates the supercharges. This is called R-symmetry.

$\rightarrow$ $SO(2,4) \times SU(4)$ is precisely the full (bosonic) symmetry of $\mathcal{N} = 4$ SYM

operators $\mathcal{O}$ in SYM $\leftrightarrow$ fluctuations of the geometry

anomalous dimensions $\leftrightarrow$ energies of KK spectrum

generating function in the field theory $\leftrightarrow$ on-shell supergravity action as a function of the boundary conditions

$$\langle \exp\left(\int \varphi\ \mathcal{O}\right)\rangle = Z_{\mathrm{string}}[\phi|_\partial = \varphi] \simeq \exp(S_{\mathrm{sugra}}[\varphi])$$

AdS/CFT is a duality relating quantum field theory to geometry.

Branes at Calabi–Yau singularities

We will be concerned with replacing $\mathcal{N} = 4$ SYM with different supersymmetric conformal field theories (SCFT) in four dimensions.

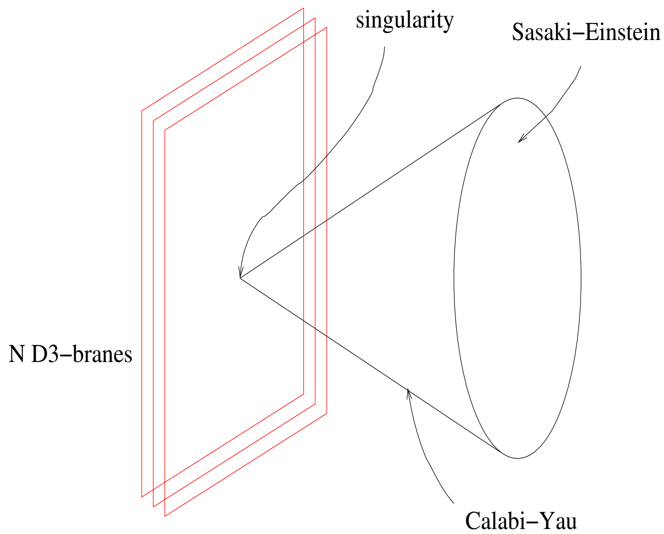
Many supersymmetric ($\mathcal{N} = 1$) gauge theories can be obtained from D3-branes as before, replacing the $\mathbb{R}^{1,9}$ ambient space with $\mathbb{R}^{1,4} \times$Calabi–Yau

Calabi–Yau: a Kähler space, equipped with a Ricci-flat metric.

CY is required in order to preserve supersymmetry. For our purposes, we need consider (non-compact) cones:

$$ds^2(\mathrm{CY}) = dr^2 + r^2 ds^2(Y_5)$$

- Consider placing N D3 branes transverse to a CY cone



The geometry, after back-reaction of the branes, is now $AdS_5 \times Y_5$.

AdS/CFT: $AdS_5 \times Y_5$ is dual to an $\mathcal{N} = 1$ SCFT

By definition, the compact five-manifolds Y_5 are Sasaki–Einstein.

The conifold: geometry

The conifold is the simplest example of CY, besides $\mathbb{C}^3$ and its quotients

Algebraic definition: $\{uv = zw\} \subset \mathbb{C}^4$

There is a Ricci-flat metric on the conifold:

$$\mathrm{d}s^2 = \mathrm{d}r^2 + r^2 \mathrm{d}s^2(T^{1,1})$$

- $T^{1,1}$ is the 5d Sasaki–Einstein base of the conifold

$$\begin{aligned} ds^2(T^{1,1}) = & \frac{1}{6}(\mathrm{d}\theta_1^2 + \sin^2\theta_1 \mathrm{d}\phi_1^2 + \mathrm{d}\theta_2^2 + \sin^2\theta_2 \mathrm{d}\phi_2^2) \\ + & \frac{1}{9}(\mathrm{d}\psi + \cos\theta_1 \mathrm{d}\phi_1 + \cos\theta_2 \mathrm{d}\phi_2)^2 \end{aligned}$$

The isometry group of $T^{1,1}$ is $SU(2) \times SU(2) \times U(1)$

The topology is that of $S^2 \times S^3$

- The metric also shows that $T^{1,1}$ has a fibration structure:

$$U(1) \to T^{1,1} \to S^2 \times S^2$$

$S^2 \times S^2$ is a Kähler–Einstein four-manifold

The volume of $T^{1,1}$ is $\mathrm{vol}(T^{1,1}) = \int \sqrt{g(T^{1,1})} = \frac{16\pi^3}{27 \cdot 4}$

The conifold: gauge theory

8 supersymmetries preserved $\Rightarrow \mathcal{N} = 1$ SCFT

$U(1)$ isometry $\Rightarrow$ R-symmetry

$SU(2) \times SU(2)$ isometry $\Rightarrow$ global flavour symmetry

- The central charge of the theory is

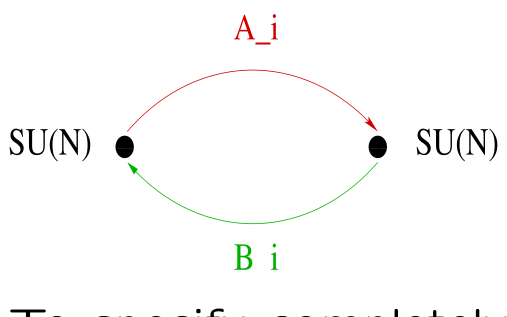
$a = \frac{27}{16} = \frac{\pi^3}{4\,\mathrm{vol}(T^{1,1})} \qquad < T^\mu_\mu >= a(\text{curvature invariant})^2$

This is true because the effective 5D action is

$$S_{5D} \sim \mathrm{vol}(T^{1,1})^{-1} \int d^5x \sqrt{g}\,(R + \dots)$$

Klebanov and Witten guessed the following gauge theory: A_i, B_i $i = 1, 2$ are bifundamental chiral fields of the gauge group $SU(N) \times SU(N)$:

	$SU(N) \times SU(N)$	$SU(2) \times SU(2)$
A_i	$(\mathbf{N}, \bar{\mathbf{N}})$	$(\mathbf{2}, \mathbf{1})$
B_i	$(\bar{\mathbf{N}}, \mathbf{N})$	$(\mathbf{1}, \mathbf{2})$



The quiver diagram is a simple graphical representation of this information

To specify completely an $\mathcal{N} = 1$ supersymmetric theory one needs also the superpotential W. Here this is

$$W = \epsilon^{ij}\epsilon^{kl}\mathrm{Tr}\, A_i B_k A_j B_l$$

I will come back to $\mathcal{N} = 1$ quiver gauge theories later.

Some Sasaki–Einstein geometry

The AdS/CFT correspondence allows us to study a large number of supersymmetric gauge theories in terms of Sasaki–Einstein (SE) geometry.

Recall, a Sasaki–Einstein manifold Y can be defined as such that its metric cone

$$ds^2_{\mathcal{C}} \;=\; dr^2 + r^2 ds^2(Y)$$

is Kähler (S) and Ricci-flat (E) = Calabi-Yau cone

From this, it follows that there exists a special object, that distinguishes different Sasaki–(Einstein) structures:

$$K = J \cdot (r\tfrac{\partial}{\partial r}) = \tfrac{\partial}{\partial \psi'}$$

This is a (unit-norm) Killing vector, called Reeb vector.

A Sasaki–Einstein manifold, can be equivalently defined as such that it admits Killing spinors ϵ, satisfying

$$\nabla_m \epsilon = \frac{i}{2}\gamma_m \epsilon$$

This is the reason why Sasaki–Einstein preserves supersymmetry in Type IIB supergravity.

Global properties of SE manifolds

• Using the fact that $\partial/\partial\psi'$ is Killing, a generic SE metric can be written as

$$ds^2(\text{SE}) = ds_4^2 + (d\psi' + \sigma)^2$$

ds_4^2 a Kähler-Einstein metric, with Kähler form $J_4 = d\sigma/2$ and a $(2,0)$ form Ω_4, such that $d\Omega_4 = i\sigma \wedge \Omega_4$.

According to the global properties of K and (ds_4^2, J_4), there are 3 types of SE manifolds:

1) Regular: (ds_4^2, J_4) is a KE manifold

These are classified in $d = 5$: S^5, $T^{1,1}$, $k\#S^2\times S^3$ $3 \leq k \leq 8$

2) Quasi-regular: (ds_4^2, J_4) is a KE orbifold

3) Irregular: (ds_4^2, J_4) is not globally defined

• Only in cases 1) and 2) the SE can be defined as $U(1)$ bundle over the respective KE base manifold/orbifold. E.g.

$$U(1) \rightarrow \quad S^5 \quad \rightarrow \mathbb{C}P^2$$

• In case 3) one cannot quotient by $\partial/\partial\psi'$: the orbits of the Reeb vector don't close, but densely fill a torus.

No examples existed before $Y^{p,q}$.

The $Y^{p,q}$ metrics

In February 2004 we found an infinite family of 5d Sasaki–Einstein metrics [Gauntlett,DM,Sparks,Waldram]-2, labeled by 2 integers p and q:

$$\begin{aligned} ds^2(Y^{p,q}) &= \frac{1-y}{6}(d\theta^2+\sin^2\theta d\phi^2)+\frac{1}{w(y)v(y)}dy^2 \\ &+ \frac{v(y)}{9}[d\psi-\cos\theta d\phi]^2+w(y)\,[d\alpha+A]^2 \end{aligned}$$

The functions $w(y,b)$, $v(y,b)$ depend on p and q through the parameter

$$b = \tfrac{1}{2} - \tfrac{p^2-3q^2}{4p^3}\sqrt{4p^2-3q^2}$$

The metrics have $SU(2)\times U(1)\times U(1)$ isometry.

And the topology of $S^2\times S^3$.

The volume is

$$\mathrm{vol}(Y^{p,q}) = \frac{q^2[2p+(4p^2-3q^2)^{1/2}]}{3p^2[3q^2-2p^2+p(4p^2-3q^2)^{1/2}]}\pi^3$$

and is generically quadratic irrational.

These were the first examples of infinite families of explicit Sasaki–Einstein metrics.

• Many new tests of the AdS/CFT correspondence could be done... provided one finds the dual gauge theories.

Calabi–Yau cones $C(Y^{p,q})$

To uncover the dual field theory, it is convenient to study the geometry of the associated Calabi–Yau cones, and in particular their singularities.

Once again, remember

$$ds^2(C(Y^{p,q})) = dr^2 + r^2 ds^2(Y^{p,q})$$

are singular Calabi–Yau cones. We will find that the singularity is toric.

We can characterise (locally) a Calabi–Yau in terms of the Kähler form J and the holomorphic $(3,0)$ form Ω

$$\text{Calabi–Yau} \quad \equiv \quad \{dJ = d\Omega = 0\}$$

For CY cones these are obtained from the SE data J_4, Ω_4, σ

$$J = r^2 J_4 + r\mathrm{d}r \wedge (\mathrm{d}\psi' + \sigma)$$
$$\Omega = e^{i\psi'} r^2 \Omega_4 \wedge \left[\mathrm{d}r + ir(\mathrm{d}\psi' + \sigma)\right]$$

J and Ω can be written down explicitly for the Calabi–Yau cones $C(Y^{p,q})$ and are useful in calculations.

Supersymmetric submanifolds

Let me pause to recall that inside the SE manifolds, there are certain supersymmetric 3-submanifolds Σ_a which play a distinguished role in the AdS/CFT correspondence.

They correspond to "baryonic operators" arising from branes wrapping them.

Taking a cone over these Σ_a we obtain 4-submanifolds in the CY $\rightarrow$ these are the divisors and are nicely characterised in terms of calibrations:

$1/2 J \wedge J$ is the calibrating form for Kähler 4-submanifolds (divisors) in the CY:

$\frac{1}{2} J \wedge J|_\xi \leq \text{vol}_\xi \quad \forall \quad$ tangent planes $\xi \subset T(\text{CY})$

$\frac{1}{2} J \wedge J|_D = \text{vol}(D) \quad \Rightarrow \quad D$ are divisors

In the $C(Y^{p,q})$'s there are four such divisors

$D_a = C(\Sigma_a) \quad \Rightarrow \quad \Sigma_a$ are susy 3-submanifolds in $Y^{p,q}$
$a = 1, 2, 3, 4$

Toric symplectic manifolds

We have passed from the Sasaki–Einstein metrics to the associated Calabi–Yau's. To understand their singular nature, we have to do a little more work.

It is useful to regard $C(Y^{p,q})$ as symplectic cones:

M, J such that $dJ = 0$ is a symplectic manifold. A cone is a manifold everywhere, except at the apex.

A vector field V on M is symplectic if $\mathcal{L}_V J = d(i_V J) = 0$

If this relation can be *integrated*, $i_V J = dy$, the function y is a "Hamiltonian" for V.

• Definition of toric manifold: a $2n$-dim symplectic manifold (M, J) with n symplectic vector fields V_i admitting n Hamiltonians y_i

$$y_i : M \longrightarrow \mathbb{R}^n$$

The y_i (moment map) are natural coordinates for symplectic toric manifolds, in which the symplectic form is simply

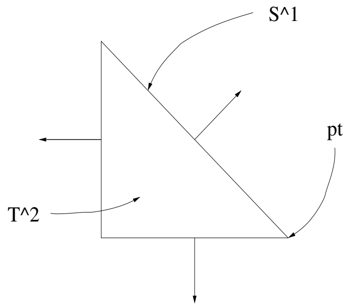
$$J = \sum_i^n dy_i \wedge d\phi^i$$

Where $\phi^i \sim \phi^i + 2\pi$ parameterize the T^n torus.

Moment polytopes and polyhedral cones

A classic result of symplectic toric geometry states that the image of a toric manifold under its moment map is a convex polytope

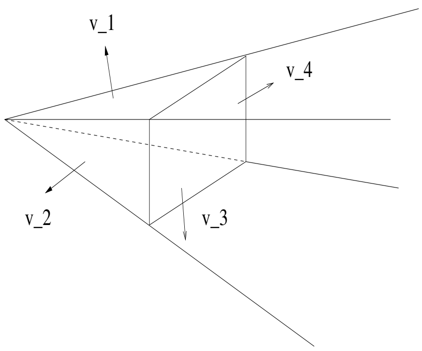
$y_i(M) = \Delta$ is a convex polytope in $\mathbb{R}^n$

Example: $y_i(\mathbb{C}P^2) =$  T^2 fibered over Δ

The S^1 collapsing over the edges is generated by the vector field specified by the normals v_a.

This result generalises to toric cones:

$y_i(M) = \mathcal{C}$ is a polyhedral cone in $\mathbb{R}^n$ with d "facets"

Example:  4-faceted $\mathcal{C}$ in $\mathbb{R}^3$

• Bottom line: toric manifolds/cones are characterised in terms of certain integral vectors $\{v_a\}$.

Toric singularities

We can now make contact with the algebraic point of view of toric geometry, which is the language in which toric singularities are usually discussed.

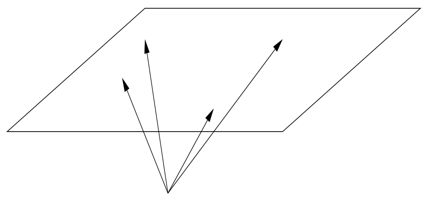
So far, we have only used symplectic properties, but we have more. The Calabi–Yau condition guarantees that the normal vectors in $v_a \in \mathbb{R}^3$ lie on plane.

This means that, by an $SL(3,\mathbb{Z})$ transformation the normal vectors can be written

$$v_a = (1, w_a)$$

where now w_a are vectors in $\mathbb{Z}^2$.

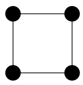
The normals v_a "end" on a plane: the information of the moment cone in $\mathbb{R}^3$ is entirely encoded in polygons in $\mathbb{R}^2$.



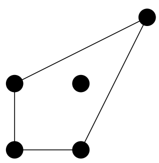
The plot of these points in $\mathbb{Z}^2$ is called the "toric diagrams".

The toric diagrams for the $Y^{p,q}$ Calabi–Yau singularities have four vertices. The location of the vertices depend on p, q. For example:

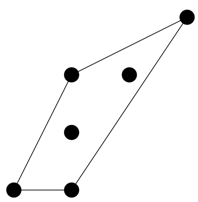
$Y^{1,0}$ (conifold) :



$Y^{2,1}$ (del Pezzo 1) :



$Y^{3,1}$:



Generalisations

We learned that $Y^{p,q}$ are the Sasaki–Einstein metrics on the base of Calabi–Yau cones with prescribed toric singularities.

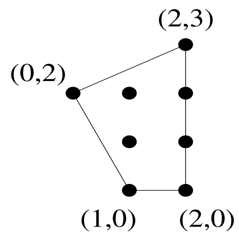
These are the most general ones, with $SU(2)$ isometry.

Given a toric singularity, how do we obtain the corresponding metric? It's very hard!

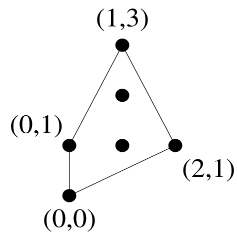
• The state-of-the-art for these metrics are the $L^{a,b,c}$ families, parameterised by 3 integers a, b, c.

Their toric diagrams contain $d = 4$ external points (facets of the polyhedral cones in $\mathbb{R}^3$). E.g.

$L^{2,6,3}$



$L^{1,5,2}$



$H(L^{a,b,c}) = \mathbb{Z}^{d-3} = \mathbb{Z} \quad \Rightarrow \quad$ SE has $S^2 \times S^3$ topology.

For these the metrics were found:

1) BPS limit of Euclidean Kerr-de Sitter black holes [Cvetic,Lu,Page,Pope]

2) scaling limit of Euclidean Demianski–Plebanski metrics [DM,Sparks]

$L^{p-q,p+q,p} = Y^{p,q}$

The $L^{a,b,a}$ singularities have been studied before under the name of "generalised conifolds": $\{uv = z^a w^b\} \subset \mathbb{C}^4$, however the metrics weren't known.

Summary of the geometry

• AdS/CFT tells us that Sasaki–Einstein 5-manifolds are important.

• Essentially S^5, $T^{1,1}$ were the only examples until recently.

• $Y^{p,q}$ provided an infinite family of examples on $S^2 \times S^3$ [Gaunlett,DM,Sparks,Waldram]-2

• The associated Calabi–Yau cones are toric [DM,Sparks]-2

• Being toric, the underlying singularities can be described in terms of vectors, or points in $\mathbb{Z}^2$, that is toric diagrams.

• Supersymmetric submanifolds are in 1-1 correspondence with the external points of these diagrams.

• The $Y^{p,q}$ metrics have been generalised

in $D = 5$: $L^{a,b,c}$ on $S^2 \times S^3$ [Cvetic,Lu,Page,Pope] [DM,Sparks]-3

in $D \geq 7$: [Gaunlett,DM,Sparks,Waldram]-3 [Chen,Lu,Pope,Vazquez-Poritz] [Cvetic,Lu,Page,Pope]

• These are all toric, in particular they have an $U(1)^{\frac{D+1}{2}}$ isometry.

Quivers gauge theories

Where are we going to find the gauge theories dual to a given toric Calabi–Yau space $\mathcal{M}$ (or equivalently, Sasaki–Einstein)?

1) $\mathcal{N} = 1$ supersymmetric gauge theories

2) the moduli space of the theory is precisely $\mathcal{M}$

3) the theory should flow to a conformal fixed point

Supersymmetric gauge theories

Field content:

Yang–Mills multiplet: A_μ, λ, D (superfield $W_\alpha \sim D_\alpha V$)

chiral multiplets (matter): X_i, ψ_i, F_i (superfield Φ)

transforming in some representation of a gauge group G.

Interactions are encoded in the complete Lagrangian:

$$S_{\text{susy}} = \int_c W^\alpha W_\alpha + W(\Phi) + c.c. + \int_{nc} \bar{\Phi} e^V \Phi$$

$W(\Phi)$ is called the superpotential and is a holomorphic function. Expanding in components, and eliminating the auxiliary fields:

$$L_{\text{susy}} = F^{\mu\nu} F_{\mu\nu} + D_\mu \bar{X}^i D^\mu X_i + \cdots + \mathcal{V}$$

The potential contains two positive definite terms

$$\mathcal{V} = D^a D_a + \bar{F}^i F_i \qquad\qquad \bar{F}^i = \frac{\partial W}{\partial X_i} \ , \quad D^a = \bar{X}^i T^a X_i$$

Supersymmetric vacua require $\mathcal{V} = 0 \ \Rightarrow$

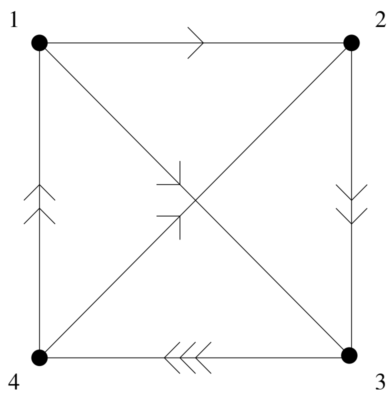
D-terms eqs: $\quad \bar{X}^i T^a X_i = 0 \qquad$ F-terms eqs: $\quad \frac{\partial W}{\partial X_i} = 0$

Quiver gauge theories

Next, we chose the gauge group to be $\underbrace{U(N) \times \ldots U(N)}_{m \text{ times}}$, and the chiral fields X_i are assigned to bifundamental representations.

The field content of these theories can be encoded into diagrams called quivers.

Consider for example the $U(N)^4$ theory with quiver diagram



Nodes $\bullet = U(N)$ gauge group

${}_i \longrightarrow{}_j$ = bifundamental X_{ij} in $(\mathbf{N}, \bar{\mathbf{N}})$ of $U_i(N) \times U_j(N)$

Multiple arrows are assigned to a representation of possible global flavour symmetry groups – here $X^\alpha_{41}, X^\alpha_{23}, X^\alpha_{34}$ are doublets of $SU(2)$.

The moduli space of quiver gauge theories is obtained giving vev's to off-diagonal fields: $U(N) \to U(1)^N$. The effective gauge group is then reduced to $G = U(1)^m$.

We also require a condition on the superpotential: W must be polynomial in X_i and each field must appear precisely twice. (Toric quiver condition).

For example, for the $U(N)^4$ theory above, this is:

$$W = \epsilon_{\alpha\beta} X_{34}^{\alpha} X_{41}^{\beta} X_{13} - \epsilon_{\alpha\beta} X_{34}^{\alpha} X_{23}^{\beta} X_{42} + \epsilon_{\alpha\beta} X_{12} X_{34}^{3} X_{41}^{\alpha} X_{23}^{\beta}$$

Finally, the moduli space is given by enforcing the D- and F-terms, modulo gauge equivalences, and it turns out (believe me) that this is precisely a 3-$\dim_{\mathbb{C}}$

Calabi–Yau toric variety

AdS/CFT guarantees that these theories flow to a superconformal fixed point.

The $Y^{p,q}$ quivers

Using the geometric information obtained from the $Y^{p,q}$ Sasaki–Einstein metrics and toric singularities, the dual toric quiver gauge theories can be conjectured.

$SU(2) \times U(1)$ flavour symmetry

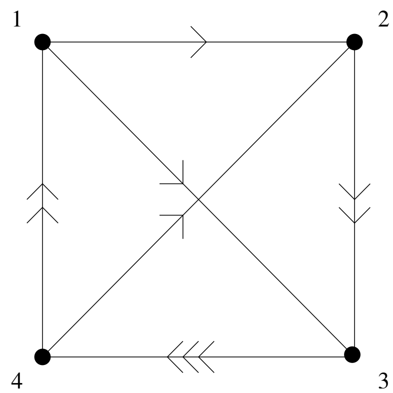
\# gauge groups = area toric diagram = $2p \Rightarrow SU(N)^{2p}$

\# bifundamental fields = $4p + 2q$ (comes from a conjectured formula)

Some already known cases:

$$Y^{p,p} = S^5/\mathbb{Z}_{2p}, \qquad Y^{p,0} = T^{1,1}/\mathbb{Z}_p, \qquad Y^{2,1} = C_{\mathbb{C}}(dP_1)$$

The quiver gauge theory for $Y^{2,1}$ was constructed some years ago:



<u>Final check</u>: compute R-charges with a-maximisation

Schematically, the bifundamental fields are

$$p\ U \text{ doublets} \quad q\ V \text{ doublets}$$

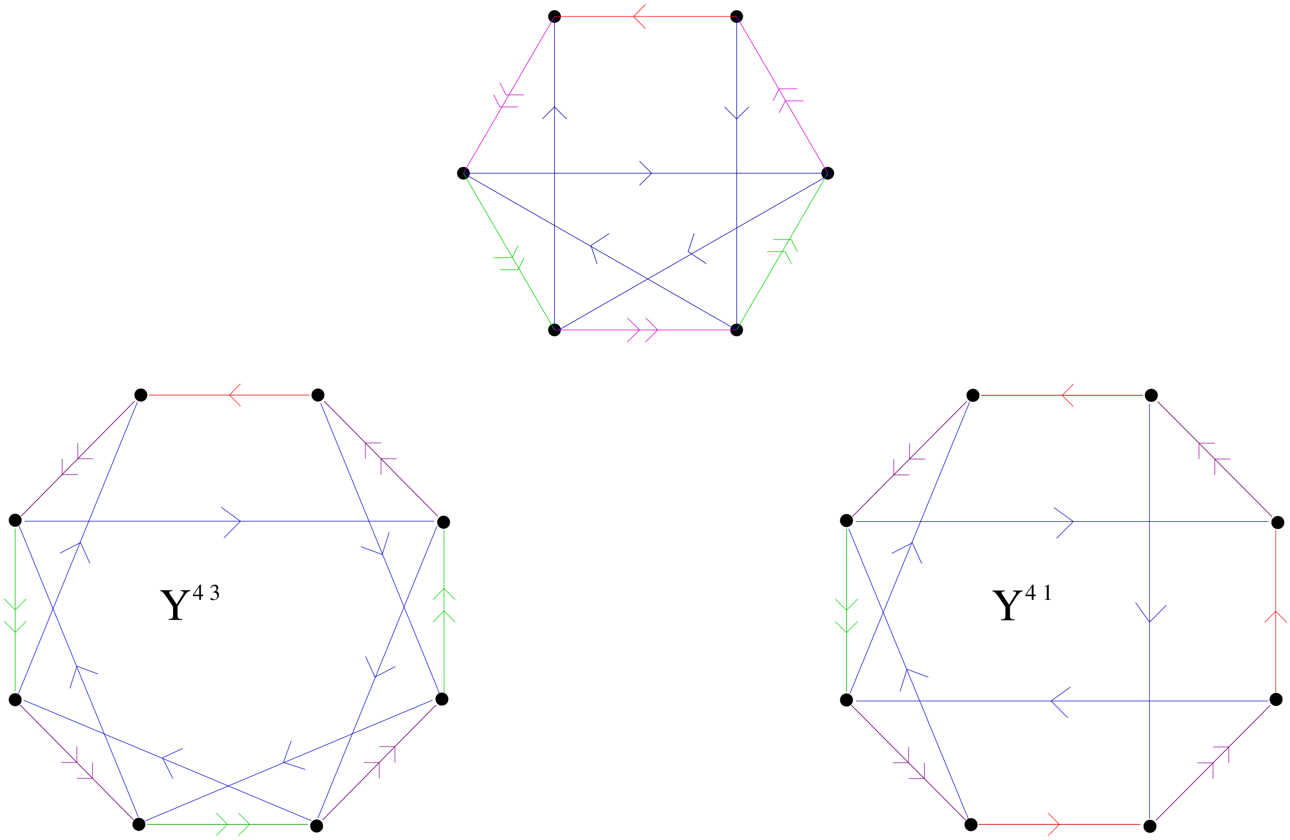
$$(p-q)\ Z \text{ singlets} \quad (p+q)\ Y \text{ singlets}$$

The superpotential is

$$W = \sum^{2q} UVY + \sum^{p-q} ZUYU$$

The fields have to be arranged into the quiver, in such a way that they respect a global $SU(2)$ flavour symmetry.

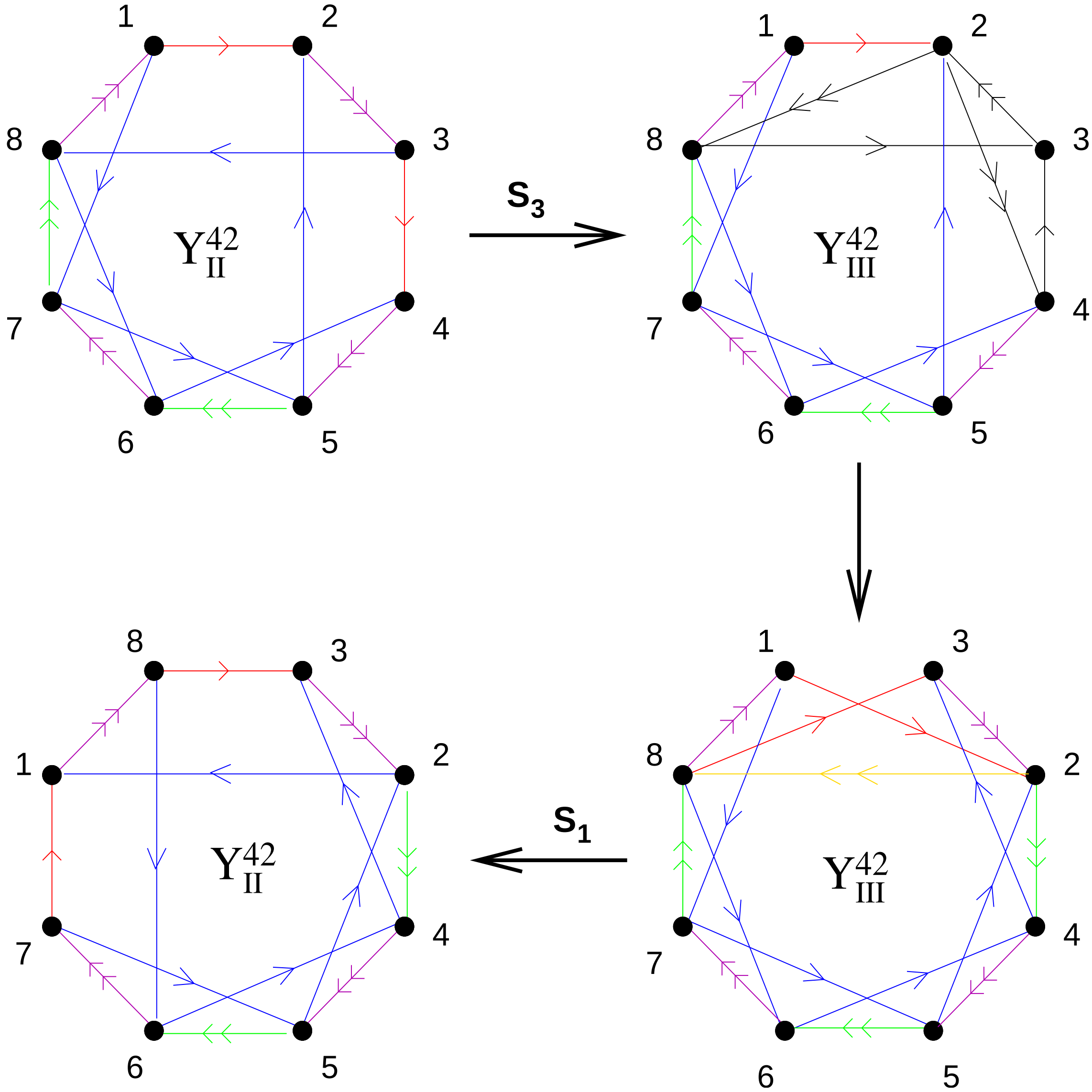
$Y^{3,2}$



These quivers are depicted in the simpler "toric phase". There is at least another presentation, which contains a new field. The key point is that the two types of quivers are related by Seiberg dualities.

Toric phases of $Y^{p,q}$ quivers

Applying Seiberg duality at a node between two Z fields creates a new field.



Two "single impurities" (Z) fuse into a "double impurity", and a new doublet C appears. [Benvenuti,Hanany,Kazakopoulos]

• Further Seiberg dualities can't create anything new: these are all connected toric phases. C fields have different R-charge, however a-maximization gives same value for a, as it should.

Baryons and susy 3-submanifolds

In the language of toric geometry, toric divisors C_a in the CY correspond to facets $\mathcal{C} \to$ external points in the toric diagram.

Toric divisor C_a in CY $\leftrightarrow$ susy 3-submanifold Σ_a in SE

D3 branes wrapped on $\Sigma_a \to$ di-baryonic operators constructed from bi-fundamentals X_a

$$\mathcal{B}[X] \quad = \quad \epsilon^{\alpha_1 \ldots \alpha_N} X_{\alpha_1}^{\beta_1} \cdots X_{\alpha_N}^{\beta_N} \epsilon_{\beta_1 \ldots \beta_N}$$

An AdS/CFT formula relates R-charges of these operators (which are the same as of X_a up to a factor of N) to volumes of susy submanifolds Σ_a

$$R_a = \frac{\pi \mathrm{vol}(\Sigma_a)}{3 \mathrm{vol}(\mathrm{SE})} \qquad a = 1, \ldots, d$$

It turns out that the charges of all bi-fundamental fields can be expressed in terms of those associated to toric submanifolds.

Let's look at $Y^{p,q}$.

$\mathcal{B}[Z]$, $\mathcal{B}[Y]$ are singlets of $SU(2)$

$\mathcal{B}[V]$, $\mathcal{B}[U]$ transform in the $\mathbf{N+1}$ of $SU(2)$

Recall there are $d = 4$ toric divisors:

$$\begin{array}{rcl} \Sigma_1 = \{y = y_1\} & \to & \mathcal{B}[Y] \\ \Sigma_2 = \{y = y_2\} & \to & \mathcal{B}[Z] \\ \Sigma_3 \simeq \Sigma_4 = \{\theta, \phi = \mathrm{const}\} & \to & \mathcal{B}[U] \\ \Sigma_1 \cup \Sigma_3 & \to & \mathcal{B}[V] \end{array}$$

$$R[V] = R[Y] + R[U]$$

$\{\theta, \phi = \mathrm{const}\} = S^2$ degeneracy $\to N+1$ degeneracy of states, from quantization of zero-modes of D3 in N units of flux [Gubser,Klebanov]

Baryons vs baryonic charges

We should distinguish between 3-cycles and supersymmetric 3-submanifolds. It is not just nomenclature.

In the toric setting:

$\#$ 3-submanifolds $=$ $\#$ external points $= d$

$\#$ 3-cycles $= \dim(H_3) = d-3$ $\qquad Y \simeq \#(d-3)(S^2 \times S^3)$

3-submanifolds $\rightarrow$ baryonic operators $\mathcal{B}_a$ $\rightarrow$ R-charges

3-cycles $\rightarrow$ baryonic symmetries $\rightarrow$ $d-3$ $U(1)_B$ charges

Let me explain better these baryonic charges.

Fluctuations of the RR 4-form potential are decomposed in a basis of harmonic 3-forms $\mathcal{H}_I$ in Y:

$$\delta C_4 = \sum_I \mathcal{H}_I \wedge A_I$$

By KK reducing this to 5D

$$\int_{\Sigma_a \times \text{time}} \delta C_4 = \sum_I Q_I[\mathcal{B}_a] \int dt A_I \quad \text{with} \quad Q_I[\mathcal{B}_a] = \int_{\Sigma_a} \mathcal{H}_I$$

The $d-3$ gauge fields A_I in AdS give rise to global symmetries in the boundary CFT. These are the baryonic $U(1)$ symmetries.

- In the toric case, the $Q_I[\mathcal{B}_a]$ can be read off from the toric diagram! They satisfy

$$\sum_a Q_I[\mathcal{B}_a]\, \vec{v}_a = 0 \quad I = 1, \ldots, d-3$$

$L^{a,b,c}$ and more toric quivers

• In the toric setting, all the bifundamental fields, with their multiplicities, R-charges, baryonic charges, and flavour charges can be extracted from the toric data, i.e. the vectors $w_i \in \mathbb{Z}^2$. [FHMSVW]

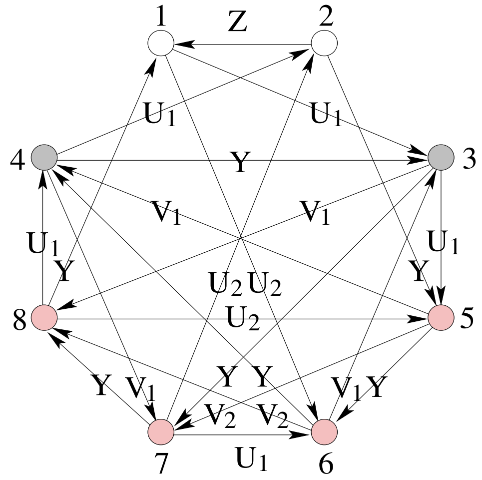
• However, it is still an open problem how to extract the specific quiver diagram, and in particular the superpotential.

• In the case of the $L^{a,b,c}$ singularities (most general toric singularities with four external points in the diagram) it has been possible to obtain the complete information on the quiver gauge theories by using the technique of dimers or brane tilings. [Benvenuti,Kruczenski], [FHMSVW], [Butti,Forcella,Zaffaroni]

Every $L^{a,b,c}$ quiver gauge theory has generically six types of bifundamental fields: Y, Z, U_1, U_2, V_1, V_2.

Although it's not possible to give an "analytic expression" for the quiver diagrams, as for $Y^{p,q}$, it is possible to write down every quiver, by combining some "building blocks".

This is for example the $L^{1,7,3}$ quiver gauge theory.



Dimers/brane tilings

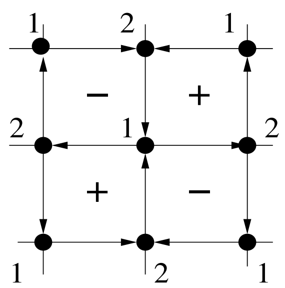
Dimer models encode in a very convenient way, the complete information about a toric quiver gauge theory, as well as the toric data. [Hanany,Kennaway] [HK+Franco,Vegh,Wecht] However, the connection is to some extent still conjectural.

These are the essential ideas involved:

• it is possible to define a periodic quiver diagram, which encodes the interactions, as well as the superpotential.

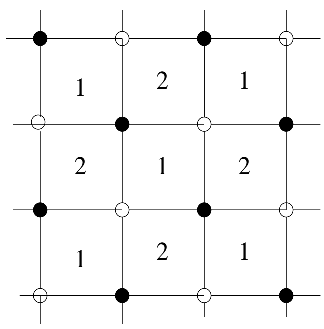
- closed loops of arrows in the period quiver $\leftrightarrow$ superpotential W term

- orientation of the loop $\leftrightarrow$ sign of the term



• The tiling/dimer is the dual graph to the periodic quiver

clockwise loop $\rightarrow$	black node $\bullet$	$-$ve W term
anticlockwise loop $\rightarrow$	white node $\circ$	$+$ve W term
nodes $\bullet$ $\rightarrow$	faces $\Diamond$ (tiles)	gauge group
edge $\longrightarrow$ $\rightarrow$	edge $\circ\!\!-\!\!\!-\!\!\bullet$	bifundamental



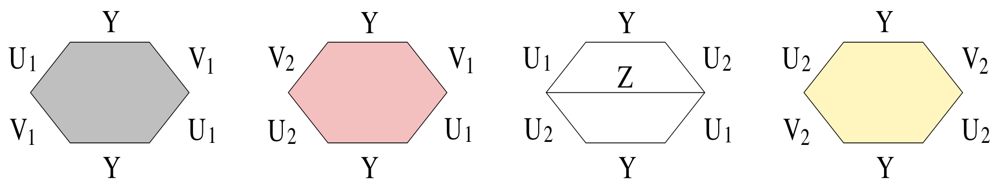
There is a recipe to compute the toric diagram of the corresponding singularity directly from a dimer!

$$\det\begin{pmatrix} & \bullet & \bullet & \bullet & \bullet & \cdots \\ \circ & 1 & 0 & w & z & \cdots \\ \vdots & z^{-1} & \cdots & & & \\ \vdots & & & & & \end{pmatrix} = \sum_{i,j} c_{ij} z^j w^i$$

$$\Rightarrow \quad \text{pairs } (i,j) \sim \text{toric diagram.}$$

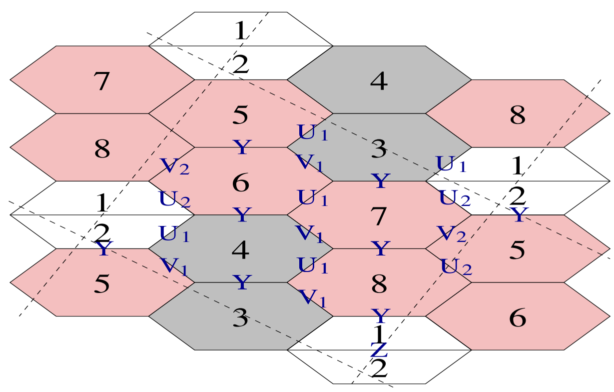
Returning to $L^{a,b,c}$, it is clearer what I meant by constructing the quivers using some "building blocks".

There are four building blocks, obeying simple gluing rules:



(A) (B) (C) (D)

With these, for any given $L^{a,b,c}$, one constructs a tiling. E.g. for $L^{1,7,3}$:



The quiver of the previous slide is just the dual of this.

a-maximisation

Any $\mathcal{N}=1$ SCFT in $d=4$ has a superconformal algebra of symmetries $SU(2,2|1)$, with bosonic part $SO(4,2)\times U(1)_R$. In particular they possess a conserved $U(1)_R$ R-symmetry.

The full symmetry group of a SCFT may contain additional global symmetries ("non-R") $\mathcal{F} = U(1)^k\times$non Abelian.

The central charges of the theory can be determined if the R-charges of all fields are known [Anselmi et al.]

$$a = \frac{3}{32}(3\text{Tr}R^3 - \text{Tr}R) \qquad c = \frac{1}{32}(9\text{Tr}R^3 - 5\text{Tr}R)$$

$$< T^\mu_\mu >= c(\text{Weyl})^2 - a(\text{Euler})^2$$

In practice, these traces look like:

$$\text{Tr}R^\alpha = \text{dim}(G) + \textstyle\sum_i n_i \text{dim}(\mathbf{r}_i)(R_i - 1)^\alpha$$

• Problem: the symmetries of the theory and the constraint that the superpotential has R-charge 2 are not sufficient to determine the R-charges. "The abelian part of $\mathcal{F}$ – F_I – can mix with $U(1)_R$."

Resolution [Intriligator,Wecht]: the exact R-charges must satisfy

$$\begin{aligned} 9\text{Tr}R^2F_I &= \text{Tr}F_I \\ \text{Tr}RF_IF_J &< 0 \end{aligned}$$

Introducing a "trial" R-symmetry

$$R_{\text{trial}} = R_0 + \sum_I s^I F_I \qquad s^I \in \mathbb{R}$$

where R_0 is a fiducial R-symmetry, this is equivalent to requiring that the central charge a_{trial} as a function of the s^I is maximised.

NB: this trick reformulates the result as a variational problem.

• The R- and a-charges of any $\mathcal{N}=1$ SCFT in 4d are algebraic numbers!

According to the AdS/CFT correspondence, the volume of the Sasaki–Einstein manifold Y_5 is related to the central charge $a(Y_5)$ of the dual SCFT via

$$\frac{\text{vol}(Y_5)}{\pi^3} = \frac{N^2}{4a(Y_5)}$$

• $\text{vol}(Y_5)/\pi^3$ must be an algebraic number and may be determined by solving a finite-dimensional extremal problem. This is a non-trivial statement for the geometry!

Applying a-maximisation to the $Y^{p,q}$ and $L^{a,b,c}$ quiver theories one finds analytic expressions for a, which agree with the values obtained from the volume . For instance, for $Y^{p,q}$ [Bertolini,Bigazzi,Cotrone], [BFHMS]

$$a_{\text{max}}(Y^{p,q}) = \frac{3p^2[3q^2-2p^2+p(4p^2-3q^2)^{1/2}]}{4q^2[2p+(4p^2-3q^2)^{1/2}]}N^2$$

Some geometry background

We wish to formulate a variational problem that determines the volume of toric Sasaki–Einstein manifolds in terms of the toric data.

$\rightarrow$ need first to declare what is the space over which the variational problem will be defined $\rightarrow$ Sasakian metrics.

<u>Definition</u>: Y_5 is Sasakian iff its metric cone is Kähler.

The Reeb: there is a canonical unit-norm Killing vector

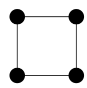
$$K = J(r\partial/\partial r)$$

Locally the metric is $ds^2(Y_5) = ds_4^2(\text{Kähler}) + (d\psi + \sigma)^2$

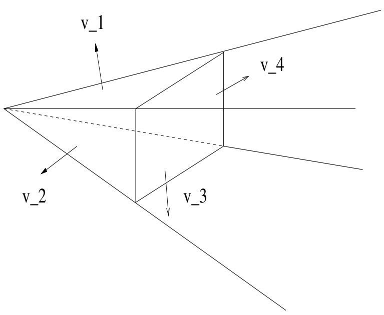
Y_5 is also Einstein iff the metric cone is Ricci-flat (CY), and we recover the earlier discussion on Sasaki–Einstein manifolds.

We consider toric Sasakian manifolds: these descend from toric Kähler cones.

Note, the description of toric singularities presented earlier is still valid, because there I assumed only the existence of a symplectic (Kähler) structure and the topological-Calabi–Yau condition that $v_a = (1, w_a) \leftrightarrow c_1(C) = 0$.

Start from a given toric diagram, e.g. $\underset{\text{(conifold)}}{Y^{1,0}}$: 

or, more conveniently, the corresponding 3D picture:



$$\mathcal{C} = \{y_i \in \mathbb{R}^3| \quad y_i\, v_a^i \leq 0 \;\; a = 1, .., d\}$$

Z-minimisation

• We use symplectic coordinates (y_i, ϕ_i), defined by $J = d\phi_i \wedge dy_i$.

In these coordinates the Kähler metric on $C(Y)$ takes the canonical form

$$ds^2(C(Y)) = G_{ij} dy_i dy_j + G^{ij} d\phi_i d\phi_j$$

$G_{ij}(y) = \frac{\partial^2 G(y)}{\partial y_i \partial y_j}$ $\rightarrow$ there is a symplectic potential $G(y)$

There exists a canonical Kähler metric specified by the potential

$$G^{\text{can}}(y) \quad = \quad \frac{1}{2}\sum_a^d l_a(y) \log l_a(y) \qquad l_a(y) \equiv y_i v_a^i$$

however, this is not Ricci-flat.

• The key object to consider is the Reeb vector. We decompose it in its $U(1)^3$ components:

$$K = b_i \frac{\partial}{\partial \phi_i} \qquad b_i = 2G_{ij} y_j \in \mathbb{R}^3$$

<u>Result</u>: any symplectic potential is written as

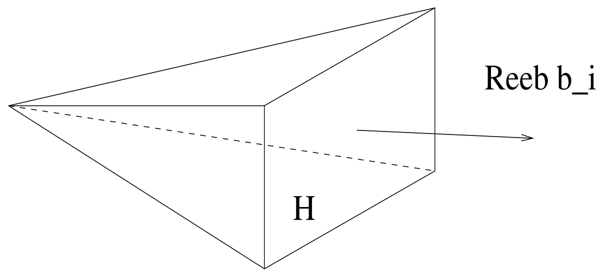
$$G(y) \quad = \quad G^{\text{can}}(y) + G_b(y) + g(y)$$

$2G_b(y) \equiv l_b(y) \log l_b(y) - l_\infty(y) \log l_\infty(y)$, $l_\infty(y) = \sum_a v_a^i y_i$

and $g(y)$ is a homogeneous function of degree 1 in y_i.

The problem has split in a finite dimensional one (determined by the Reeb vector) and an infinite dimensional part (the function g). However, it turns out that this g is irrelevant.

• Combine $\vec{v}_a$ with $\vec{b}$ to form a finite polytope Δ_b in $\mathbb{R}^3$



• The crucial point is that integrals in Y are reduced to integrals in Δ_b. E.g.

$$\mathrm{vol}(Y) = 6(2\pi)^3 \mathrm{vol}(\Delta_b) \qquad \mathrm{vol}(\Sigma_a) = 4(2\pi)^2 \frac{1}{|v_a|} \mathrm{vol}(\mathcal{F}_a)$$

NB: these depend only on b.

• Finally, take the Einstein–Hilbert action on Y:

$$S[g_Y] = \int_Y (R + \Lambda)$$

Using the above formalism, and the general expression for $G(y)$, S restricted to the space of Sasakian metrics will turn out to be a function of the Reeb vector only!

After some work.. $\quad S[g_Y] \propto (b_1 - 2)\mathrm{vol}(\Delta_b) \equiv Z[b]$

$$\frac{\partial Z}{\partial b_i} = 0 \quad \Rightarrow \quad \text{Sasaki–Einstein metric}$$

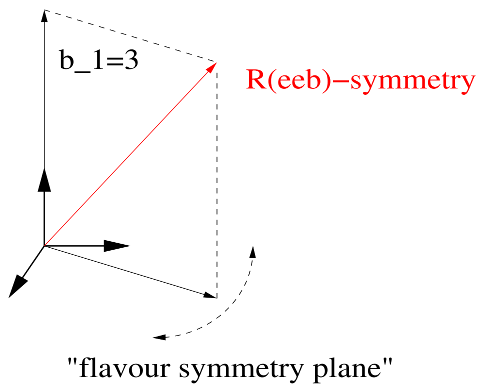
Some comments

• Plugging the Reeb vector so determined back into the formula for the volumes, we obtain the volumes of Sasaki–Einstein metrics for an arbitrary toric singularity, without actually knowing the metric.

• It can be proved that there always exists a unique minimum.

• These volumes are algebraic numbers. Namely, these are roots of algebraic equations with rational coefficients.

• The minimisation actually takes place over 2 (in general n-1) variables (it's easy to show that $b_1 = 3$ is fixed)



• Z-minimisation applies to all dimensions $\rightarrow$ in $D = 7$ implies using AdS/CFT that the R-symmetry of 3D SCFT's must be determined by some variational problem $\rightarrow$ τ_{RR}-minimisation! [BGISW]

• For the case of physical interest $n = 3$, there is an explicit formula for the volumes and Z in terms of 3×3 determinants and the toric data. E.g.

$$\text{vol}(Y) = \frac{\pi^3}{b_1} \sum_a^d \frac{(v_{a-1}, v_a, v_{a+1})}{(b, v_{a-1}, v_a)(b, v_a, v_{a+1})}$$

Using this, it is straightforward to compute the volumes of the $Y^{p,q}, L^{a,b,c}$ manifolds, or in fact of any arbitrary toric Sasaki–Einstein manifold.

Z-minimisation versus a-maximisation

• The two methods always give the same result in all the (infinitely many) examples that can be checked. We can explain this remarkable agreement more precisely.

• First, notice that the functions a and Z appear to depend on different variables, and their number also seems different.

a is a cubic function of all the non-R flavour and baryonic symmetries, which are arbitrarily many

Z is a rational function of two components of the Reeb vector

• We need to understand the flavour and baryonic symmetries implied by the geometry.

Recall, there are $d-3$ baryonic $U(1)_B$'s coming from the topologically independent 3-cycles.

There are 3-1=2 flavour $U(1)_F$'s coming from the T^3 isometry. More precisely, there are 2 Killing vectors $V_i \in T^3$ that leave the spinors (or the $(3,0)$-form) invariant

$$\mathcal{L}_{V_i}\psi = 0 \quad i = 1,2$$

• In summary, any SCFT dual to a toric SE manifold, has

$$U(1)_F^2 \times U(1)_B^{d-3} \text{ global symmetries}$$

$\Rightarrow$ a_{trial} is a function of $d-1$ variables

A formula for a_{trial}

The detailed map from a given toric geometry to the corresponding field theory is still missing. However, the key point is that in order to compare to Z-minimisation, one only needs a_{trial}.

It turns out that this can be written down in general, without detailed knowledge of the quiver or superpotential.

• We can define a geometric phase (among all the Seiberg dual "toric phases") of the gauge theory, as containing the minimal number of fields.

Among these fields there are d distinguished ones, associated to the toric divisors, i.e. the supersymmetric sub-manifolds Σ_a.

$$\Sigma_a \leftrightarrow \qquad \mathcal{B}[X_a] = \epsilon^{\alpha_1 \dots \alpha_N} X^{\beta_1}_{\alpha_1} \cdots X^{\beta_N}_{\alpha_N} \epsilon_{\beta_1 \dots \beta_N} \qquad \leftrightarrow X_a$$

$\rightarrow$ All the R-charges are parameterized by these d geometric R-charges. Moreover, there is a relation among them [MSY]

$$\textstyle\sum_a^d R_a = 2 \qquad \text{with} \qquad R_a = \frac{\pi \text{vol}(\Sigma_a)}{\text{vol}(Y)}$$

$\Rightarrow$ There is an analytic formula for a_{trial} in terms of $d-1$ variables, as expected.

$$\frac{32}{9} a_{\text{trial}}(R_a) = N_g + \frac{1}{2} \sum_{a=1}^{d} \sum_{b=a}^{a-1} m_{a,b} \left[\left(\sum_{c=a}^{b} R_c \right) - 1 \right]^3$$

Where $m_{a,b} = (w_{b+1} - w_b) \times (w_a - w_{a-1})$. [BZ] and [MS]-unpublished following [FHMSVW].

A puzzling relation

Using the AdS/CFT formula, relating R-charges to volumes of susy submanifolds Σ_a, we get the geometric R-charges as functions of the Reeb vector [MSY]

$$R_a(b_2, b_3) = \frac{\pi \mathrm{vol}(\Sigma_a)}{\mathrm{vol}(Y)}(b_2, b_3) \qquad a = 1, \ldots, d$$

<u>Note</u>: this is not really an AdS/CFT formula, if used with an arbitrary Reeb vector b. It should be valid only for the Sasaki–Einstein point.

- Using this relation, a_{trial} becomes now a function of b_2, b_3. Not only! It tuns out that

$$32 a_{\mathrm{trial}}(b_2, b_3) = \frac{1}{Z(b_2, b_3)}$$

before extremisation! This has been proven by [BZ] (also noticed by us [MS] in unpublished work).

It explains why the two problems give the same result. :-)

However, it is not clear why this relation holds. :-(

Summary

The construction of the infinitely many AdS/CFT dual pairs $Y^{p,q}$ (before only only S^5 and $T^{1,1}$ were known) has produced a lot of progress in different areas: SE geometry, construction of gauge theories for branes at singularities, generalizations of a-maximisation, etc.

• Several explicit families of Sasaki–Einstein metrics have been constructed in 5 and higher dimensions: $L^{a,b,c}$, etc.

• Z-minimisation $\rightarrow$ compute volumes of Sasaki–Einstein metrics $\Rightarrow$ R-charges in the dual SCFT.

• Z-minimisation exists in arbitrary dimension. In $D = 3$ it predicted, using AdS/CFT, that the R-charges of 3D $\mathcal{N} = 2$ SCFT must be obtained in a fashion similar to a-maximisation (but here there is no a!). Subsequently, Barnes et al. have discovered τ_{RR}-minimisation.

• New infinite families of quiver gauge theories have been constructed (before, a few examples were known, e.g. del Pezzo's). E.g. $X^{p,q}$ – whose dual metrics are unknown.

• A lot of information on the dual gauge theory can be obtained from the geometry, without the explicit metric. I.e. all bifundamentals with their global quantum numbers.

• Progress towards a 1-1 correspondence between toric singularities and $\mathcal{N} = 1$ quiver gauge theories. E. g. brane tilings.

Outlook and open problems

• It would be nice to make the recipe for extracting the gauge theory from a toric singularity more rigorous and complete. E.g. understand more rigorously the dimers.

• Although remain details to fill in, the AdS/CFT correspondence in the toric sector is completely checked. What about non-toric or in fact, non Sasaki–Einstein geometry?

Can we construct more general explicit Sasaki–Einstein metrics?

How does Z-minimisation generalise to these classes of geometries? In particular prove that $\mathrm{vol}(Y_5)/\pi^3$ are always algebraic numbers. It may be a difficult problem in geometry.

How do we extract the gauge theory from a non-toric singularity? And for a general $AdS_5 \times Y_5$ geometry with Y_5 not Sasaki–Einstein?

Presumably, infinite families of duals are sitting there, in IIB, as well as in 11D. This territory is almost completely unexplored!

• The role of Sasakian geometry. Somewhat strangely, the relation $32aZ = 1$ is true "off-shell". This is not expected to hold, because Sasakian metrics (not Einstein) do not appear to solve any known equation of motion. Perhaps this is related to deformations of AdS, e.g. flows.