Geometry of Higher-Dimensional Black Hole Thermodynamics

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Abstract

We apply the Ruppeiner theory to black hole thermodynamics in higher dimensions and obtained interesting results. We think this may be a justification for applying this theory to black hole solutions that arise from various gravity theories, *e.g.* String Theory.

Plan of talk

- Thermodynamics as Geometry
- Reissner-Nordström (RN) Black Hole
- Kerr Black Hole
- Multiple-spin Kerr Black Hole
- Summary

1. *Thermodynamics as Geometry*

George Ruppeiner: Phase transitions & Critical Phenomena may be approached thermodynamically by Riemannian geometry, with a metric related to thermodynamic fluctuations.

Take a Hessian matrix of thermodynamic entropy and define it as a metric on the state space

$$g^R_{ij} = -\frac{\partial^2 S(X)}{\partial X^i \partial X^j}, \quad X = X(M, N^a)$$

M the mass and N^a the extensive parameters of the system.

- Known as the *Ruppeiner metric*.
- g^R_{ij} can take any dimension.
- Most commonly studied Ruppeiner metrics so far are the 2×2 metrics.
- If g^R_{ij} is flat, then we have a system with no underlying statistical mechanical interactions, $e.g.$ the ideal gas.
- If g^R_{ij} is non-flat and its curvature has singularity(ties), we have a signal of critical phenomena.

There is a dual metric to the Ruppeiner metric, it is known as the Weinhold metric (Frank A. Weinhold 1975). It is the Hessian of the mass (internal energy) defined as

$$g_{ij}^{W} = \partial_i \partial_j M(S, N^a)$$

N^a being any other extensive variables. The two metrics are conformally related to each other via

$$ds^2 = g_{ij}^{R} dM^i dM^j = \frac{1}{T} g_{ij}^{W} dS^i dS^j$$

Temperature is given by

$$T = \frac{\partial M}{\partial S}$$

- Ruppeiner theory has been successful and received support from various directions (e.g. Salamon, et al in J. Chem. Phys, vol 82, 5. 2413 (1982))
- Hawking & Bekenstein: Black holes are thermodynamic systems.$S = S(M, J, Q)$

- Ruppeiner theory has been applied to black holes (*hep-th/9803261*, *gr-qc/0304015*)
- Results so far have been as anticipated, *i.e.* for simple black hole solutions we have flat Ruppeiner geometry and *vice versa*.
- In 2+1, the BTZ black hole has a flat Ruppeiner metric.
- There are results in adS space, *e.g.* the RN-adS where Hawking-Page transition complicates the geometry.

2. Reissner-Nordström Black Hole

The entropy of RN (after redefinition of k_B, G and $\hbar = 1$) reads

$$S = \left(M + M\sqrt{1 - \frac{d-2}{2(d-3)}\frac{Q^2}{M^2}} \right)^{\dfrac{d-2}{d-3}}$$

Inversion of this eq gives

$$M = \frac{S^{\frac{d-3}{d-2}}}{2} + \frac{d-2}{4(d-3)}\frac{Q^2}{S^{\frac{d-3}{d-2}}}$$

Taking the Hessian of M, we get the Weinhold metric, after diagonalization reads

$$ds_W^2 = S^{-\frac{d-1}{d-2}}\left[-\frac{1}{2}\frac{d-3}{(d-2)^2}(1-u^2)dS^2 + S^2 du^2 \right]$$

using

$$u = \sqrt{\frac{d-2}{2(d-3)}}\frac{Q}{S^{\frac{d-3}{d-2}}}$$

By conformal transformation

$$ds_R^2 = \frac{-dS^2}{(d-2)S} + \frac{2(d-2)S}{(d-3)} \frac{du^2}{1-u^2}$$

It is a flat metric. Introducing new coordinates

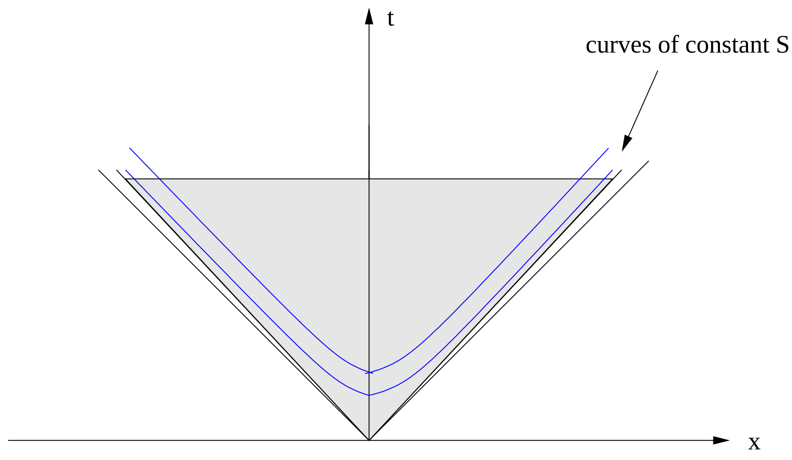
$$\tau = 2\sqrt{\frac{S}{d-2}} \quad \text{and} \quad \sin\frac{\sigma\sqrt{2(d-3)}}{d-2} = u$$

we get the Ruppeiner metric in Rindler coordinates as

$$ds^2 = -d\tau^2 + \tau^2 d\sigma^2$$

with

$$-\frac{d-2}{2\sqrt{2(d-3)}}\pi \leq \sigma \leq \frac{d-2}{2\sqrt{2(d-3)}}\pi$$



If we use

$$t = \tau \cosh \sigma \quad \text{and} \quad x = \tau \sinh \sigma$$

we obtain a Rindler wedge with an opening angle depending on d:

$$\tanh -\frac{(d-2)\pi}{2\sqrt{2(d-3)}} \leq \frac{x}{t} \leq \tanh \frac{(d-2)\pi}{2\sqrt{2(d-3)}}$$

For $d = 4$ we get the wedge as seen in figure. Curves of constant S are given by $S = \frac{1}{2}(t^2 - x^2)$. Note that the opening angle of the wedge of the RN black hole grows as $d \to \infty$.

4. Kerr Black Hole

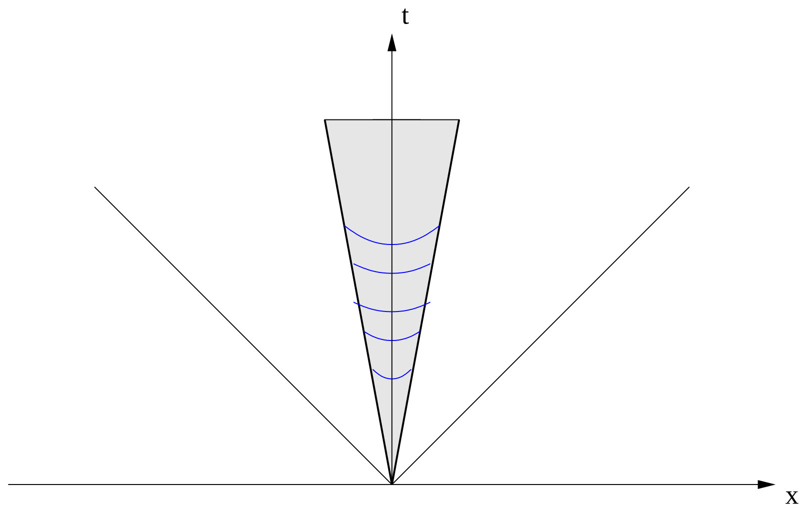
Kerr black hole = uncharged spinning black hole. In $d > 4$ we can have more than one angular momentum. Do the single-spin case in any d.

- Cannot solve for r_+ in any d but can work with the Weinhold metric. The mass of the Kerr black hole in arbitrary d is given by

$$M = \frac{d-2}{4} S^{\frac{d-3}{d-2}} \left(1 + \frac{4J^2}{S^2}\right)^{1/(d-2)}$$

Results:

- Weinhold metric $g_{ij}^W = \partial_i \partial_j M(S, J)$ can be worked out. It is a flat metric.
- Can be transformed into Rindler coord
- Wedge of state space with specific opening angle for $d = 4, 5$.
- Special feature: for $d \geq 6$ the wedge fills the entire light cone because there are no extremal limits for Kerr black hole in $d \geq 6$.
- Ruppeiner geometry is curved and has curvature blow-up in all dimensions.



- Curvature scalar is singular at extremal limit for $d = 4, 5$. For $d \geq 6$ it is divergent along the curve (that depends on the dimensionality), also found by Emparan and Myers (*hep-th/0308056*) to be where the Kerr black hole becomes unstable and changes behavior to be like a black membrane.

5. Multiple-Spin Kerr Black Hole

The Kerr black hole in $d \geq 5$ can have more than one angular momentum.

- Motivation: see if there is any chance it would simpler than in Kerr-Newman (KN) case.
- Pick the Kerr in $d = 5$ with double spins (3-parameter problem)

$$M = M(S, J_1, J_2)$$

- Weinhold and Ruppeiner geometry are curved $\Rightarrow$ not simpler than KN.
- Both the Weinhold and Ruppeiner curvatures have divergences in the extremal limit of the double-spin Kerr black hole in $d = 5$. Similar to the Kerr-Newman black hole (in $d = 4$).
- Calculations in 3×3 problems need labor of computers! We used CLASSI (free program distributed by Jan E. Åman) and GRTensor for Maple

We seek explanation for flatness condition.

- Mathematical explanation for flatness condition:

$$\psi(x,y) = x^a F\left(\frac{x}{y}\right), \quad a = \text{constant}$$

- RN black hole's entropy and Kerr black hole's mass have this form

Spacetime dimension	**Black hole family**	**Ruppeiner**	**Weinhold**
$d = 4$	Kerr	Curved	Flat
	RN	Flat	Curved
$d = 5$	Kerr	Curved	Flat
	double-spin Kerr	Curved	Curved
	RN	Flat	Curved
$d = 6$	Kerr	Curved	Flat
	RN	Flat	Curved
any d	Kerr	Curved	Flat
	RN	Flat	Curved
$d = 3$	BTZ	Flat	Curved
$d = 4$	RNadS	Curved	Curved

Table 1: Geometry of higher-dimensional black hole thermodynamics.

6. Summary

- To our surprise, the GEOMETRY of black hole thermodynamics in higher d is the same as that in $d = 4$
- Still cannot conclude in the ideal gas manner $\Leftrightarrow$ microstructures of black holes still are unknown
- Ruppeiner curvatures are physically suggestive in all dimensions