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The GMO sum rule

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See, V. Abaev, P. Metsä, and MS, EPJ A32 (2007) 321, hep-ph/0704.3167

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1. The GMO sum rule

πN amplitude:

$$T_{\pi N} = \bar{u}'[A(\nu,t) + \frac{1}{2}\gamma^{\mu}(q+q')_{\mu}B(\nu,t)]u,$$

$$\nu = \frac{s-u}{4M} = \omega + \frac{t}{4M}.$$

$$D(\nu,t) = A(\nu,t) + \nu B(\nu,t)$$

Optical theorem: $\mathrm{Im}\, D(\omega, t=0) = k_{LAB}\,\sigma$

Isospin: $D^{\pm} = \frac{1}{2}(D_{\pi^- p} \pm D_{\pi^+ p})$

Forward dispersion relation for D^-:

$$\mathrm{Re}\, D^-(\omega) = 8\pi f^2 \frac{\omega}{\omega^2 - \omega_B^2} + \frac{2\omega}{\pi} \mathrm{P} \int_0^\infty \frac{\sigma^-(k')}{k'^2 - k^2} \frac{k'^2\, dk'}{\omega'},$$

$$\omega_B = -\mu^2/2M.$$

Goldberger-Miyazawa-Oehme sum rule:

$$D^-(\mu) = \frac{8\pi f^2}{\mu(1 - (\frac{\mu}{2M})^2)} + 4\pi\mu J^- = 4\pi(1 + \frac{\mu}{M}) a_{0+}^-$$

where

$$J^- = \frac{1}{4\pi^2} \int_0^\infty \frac{\sigma_{\pi^- p}(k) - \sigma_{\pi^+ p}(k)}{\omega}\, dk.$$

2. Pionic hydrogen

The PSI collaboration

Level shift and width of the 1s level:

$\epsilon_{1s} = -7.120 \pm 0.008 \pm 0.009$ eV,

$\Gamma_{1s} = 0.868 \pm 0.040 \pm 0.038$ eV.

Deser formula gives (including em. corrections)

$$\epsilon_{1s} = -2\alpha^3 \, \mu_c^2 \, (a_{0+}^+ + a_{0+}^-)(1 + \delta_\epsilon),$$

where the correction factor (next-to-leading order) has the value

$$\delta_\epsilon = (-7.2 \pm 2.9) \times 10^{-2}.$$

Potential models give numbers which are very different:

$$\delta_\epsilon = (-2.1 \pm 0.5) \times 10^{-2}.$$

For the width we have

$$\Gamma = 8\alpha^3 \mu_c^2 \, q_0 \, (1 + \frac{1}{P})[a_{0+}^-(1 + \delta_\Gamma)]^2,$$

where the Panofsky ratio

$$P = \frac{\sigma(\pi^- p \to \pi^0 n)}{\sigma(\pi^- p \to \gamma n)} = 1.546 \pm 0.009,$$

and in leading order

$$\delta_\Gamma = (0.6 \pm 0.2) \times 10^{-2}.$$

Here again a potential model would give

$$\delta_\Gamma = (-1.3 \pm 0.5) \times 10^{-2}.$$

There are recent indications that the width could be much smaller (hep-ph/0610201)

$$\Gamma_{1s} = 0.823 \pm 0.019 \text{ eV}.$$

Scattering lengths

With the identification $a_{\pi^- p} = a^+_{0+} + a^-_{0+}$ we get

$$a_{\pi^- p} = 0.0933 \pm 0.0029 \,\frac{1}{\mu},$$

and

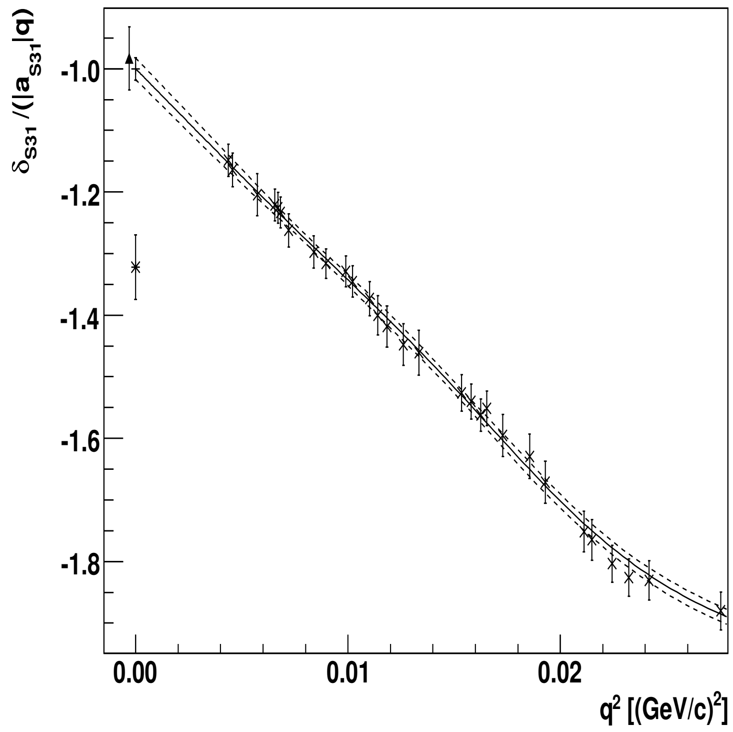
$$a^-_{0+} = 0.0888 \pm 0.0040 \,\frac{1}{\mu}.$$

The preliminary (smaller) value for Γ_{1s} would give

$$a^-_{0+} = 0.0865 \pm 0.0010 \,\frac{1}{\mu}.$$

3. The π^+p s-wave scattering length

For the $\delta_{S31}/|a_{S31}q|$ we obtain:



where the scattering length gets the value:

$$a_{S31} \equiv a_{\pi^+ p} = -0.0764 \pm 0.0014 \, \frac{1}{\mu}.$$

The result of Matsinos et al. is

$$a_{\pi^+ p} = -0.0751 \pm 0.0039 \, \frac{1}{\mu}$$

and the Karlsruhe number

$$a_{\pi^+ p} = -0.1010 \pm 0.0040 \, \frac{1}{\mu}.$$

4. The J^- integral

Input:

-$\sigma^{\mathrm{Tot}}_{\pi^\pm p}$ in the range 0.16-340 GeV/c (640 GeV/c for $\pi^- p$)

-pionic hydrogen level shift and width results

-"partial total" cross sections

-Re $D^+(t=0)$ results at low energy

-real-to-imaginary ratios

Corrections:

-Tromborg (below 0.725 GeV/c)

-P_{33} splitting for $\pi^- p$

Table 1: Contributions to J^{-} (mb) of the different high-energy ranges of the laboratory momentum k.

Input	10-350 GeV/c	350- GeV/c
Höhler (1983)	0.08786	0.01787
Donnachie-Landshoff (1992)	0.09968	0.02514
Gauron-Nicolescu (2000)	0.10665	0.02012
PDG (2006)	0.09587	-
Present work	0.09609	-

Table 2: Contributions to J^- (mb) of the low and intermediate energy ranges of the laboratory momentum k.

Input	0-2.03 GeV/c	2.03-10 GeV/c
KH80 (1980)	-1.27853	0.10691
KA84 (1985)	-1.31266	0.13802
FA02 (2004)	-1.30213	-
Present work	-1.29757	0.12046

Table 3: The values for the integral J^- (mb).

Source	J^- (mb)
Höhler-Kaiser (1980)	-1.06
Koch (1985)	-1.077 ± 0.047
Gibbs *et al.* (1998)	-1.051 ± 0.005[a]
Ericson *et al.* (2002)	-1.083 ± 0.032
Present work	-1.060 ± 0.030

[a]Statistical error only.

Error analysis

- The statistical error for J^- is 0.007 mb
- The systematic effect due to discrepant data 0.012 mb
- The effect of the coupling constant 0.001 mb
- The asymptotic behaviour, the estimated uncertainty is 0.004 mb
- Coulomb correction between 0.725 - 2.03 GeV/c 0.006 mb

5. The πN coupling constant

The πN coupling can be extracted from the sum rule

$$f^2 = \frac{1}{2}[1 - (\frac{\mu}{2m})^2] \times [\frac{1}{2}(1 + \frac{\mu}{m})(a_{\pi^- p} - a_{\pi^+ p})\mu - J^- \mu^2].$$

Inserting the values for $a_{\pi^- p}$, $a_{\pi^+ p}$ and J^- gives

$$f^2 = 0.075 \pm 0.002.$$

By invoking the isospin invariance we can relate $a_{\pi^- p} - a_{\pi^+ p} = 2\, a^-_{0+}$. The numbers from the pionic hydrogen level width measurement would give the couplings in the range $f^2 = 0.076 - 0.077$.

Other πN analyses

• VPI-GWU analysis (FA02)

R.A. Arndt et al., Phys. Rev. **C69** (2004) 035213.

pw's up to 2.1 GeV, fixed-t constraints up to 1 GeV with $-0.4\ \mathrm{GeV}^2 \leq t \leq 0.\ \mathrm{GeV}^2$,

$f^2 = 0.0761 \pm 0.0006$, $a_{\pi^- p} = 0.0856 \pm 0.0010\ \mu^{-1}$

http://gwdac.phys.gwu.edu/

• Bugg analysis

D.V. Bugg, Eur. Phys. J. **C33** (2004) 505.

$f^2 = 0.0755 - 0.0763 \pm 0.0007$, $a_{\pi^- p} = 0.0850 - 0.0863\ \mu^{-1}$

• Pionic hydrogen analysis

T.E.O. Ericson et al., Phys. Lett. **B594** (2004) 76.

$f^2 = 0.0777 \pm 0.0009$, $a_{\pi^- p} = 0.0870 \pm 0.0005\ \mu^{-1}$

Making use of the potential model electromagnetic corrections would yield

$$f^2 = 0.077 - 0.078.$$

6. Missing pieces and outlook

The uncertainty in δ_ϵ is mainly due to the largely unknown low-energy constant f_1 of the electromagnetic interaction (in ChPT).

The largest effect to the uncertainty in f^2 is due to the uncertainty in $a_{\pi^- p}$.

Final results for ϵ_{1s} and Γ_{1s} from the pionic hydrogen experiment are eagerly expected.